Ordinary and Partial Differential Equations

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CHAPTER 1

First-order differential equations

The Italian physicist Galileo Galilei famously said that the world is written in the language of mathematics. As it stands, this assertion is certainly questionable from a philosophical standpoint; however it is not controversial that the universe is *read* in mathematical language, which is to say, the physical laws governing it are formulated and studied in mathematical terms.

Whilst elementary algebra, in the form of standard equations, is sufficient to analyze most static problems, it is ill-suited to the understanding of dynamical phenomena, involving physical quantities which change over time. For those, the concept of *differentiation*, one of the foundational notions of calculus, naturally enters the picture, as it formalizes mathematically the intuition of *infinitesimal rate of change* of a certain quantity. An overwhelming majority of the laws of nature, such as Newton's law of gravitation or Maxwell's equations for electromagnetism, express the instantaneous rate of change of a given quantity of interest in terms of other variables of the problem, which might be, for instance, the time variable and the physical quantity under investigation itself. We shall see a wealth of incarnations of this general formulation, starting already with §1.1; in mathematical language, laws of this kind are expressed via *differential equations*, the core topic of this course.

1.1. Differential equations and mathematical models

1.1.1. Some introductory examples. As already alluded to in the introduction to this chapter, a differential equation is a mathematical identity relating a certain unknown function to its derivatives of higher order. For the most part of this course, we shall be interested in functions of a single real variable; differential equations involving those are called *ordinary*, as opposed to *partial differential equations*, the subject of the last chapter of this course, where the unknown is a function of several real variables and the equation relates such function to its *partial derivatives* of higher order.

Before giving the abstract definition of an ordinary differential equation, let us have a look at a few motivating examples.

Example 1.1.1. The equation

$$x'(t) = 2x(t) - t$$

involves an unknown function $x(t)$, of the independent variable t, and its derivative $x'(t)$. We shall classify this example as a *first-order differential equation*.

Example 1.1.2. The equation

$$y''(t) + y'(t) - 3y(t) = \cos t$$

features an unknown function $y(t)$ together with its first two derivatives $y'(t)$, $y''(t)$. We shall refer to it as a *second-order differential equation*.

Example 1.1.3. Consider the equation

$$y'(x) = 3x^2 y(x) \,, \tag{1.1.1}$$

which involves an unknown function $y(x)$ and its derivative $y'(x)$. Let us verify that the **one-parameter family**[1] of functions

$$y(x) = Ce^{x^3} , \tag{1.1.2}$$

where C is a constant allowed to range over all real numbers, gives an infinite set of *solutions* to the equation, namely of functions verifying the identity in (1.1.1). To this end, let's compute the derivative $y'(x)$ of $y(x)$ using the familiar chain rule:

$$y'(x) = Ce^{x^3}(3x^2) .$$

We immediately realize that the latter expression equals precisely $3x^2y(x)$, as desired.

Notice that, in the last example, we encountered *infinitely many* different functions satisfying the given equation. We shall see that this is a typical feature of differential equations[2]. For the moment, we might ask ourselves: are the functions in (1.1.2) the *only* possible solutions to (1.1.2), or are there any others? over the course of this chapter, we shall learn a variety of methods to explicitly *solve* differential equations such as the one under consideration here, which will enable us to conclude that, in this specific example, there are no solutions other than the ones in (1.1.2).

1.1.2. Mathematical modelling through differential equations. We shall now present two introductory examples of mathematical modeling of a physical phenomenon governed by a law which lends itself to a formulation via a differential equation.

Example 1.1.4 (Newton's law of cooling). In thermodynamics, *Newton's law of cooling* describes the time evolution of the temperature of an object in terms of the temperature of the surrounding environment, such as a hot rock immersed in a glass of cold water. If the surrounding medium is substantially larger than the object under scrutiny, it is physically reasonable to assume that the ambient temperature remains constant in time, namely is unaltered by the interaction with the smaller object. Newton's law of cooling then asserts that the rate of change of the temperature of the object is directly proportional to the difference between the temperature of the object and the one of the ambient space.

Let us model this phenomenon mathematically, specifically let us phrase Newton's law in mathematical terms. Let A denote the temperature of the environment. If $T(t)$ indicates the temperature of the object at time t, then its rate of change is expressed, as is well known from earlier Calculus courses, by the derivative $T'(t)$. We may thus formulate Newton's law as the differential equation

$$T'(t) = -k(T(t) - A)$$

for a certain proportionality constant $k > 0$ (which is part of the physical data of the problem). The reason for the sign of the proportionality constant, which is negative (beware the minus sign in front of the k), is of physical nature: it is well known from experimental evidence that the temperature of the object will increase if it is lower than the one of the environment ($T(t) < A$ implies $T'(t) > 0$), and decrease if it is higher ($T(t) > A$ implies $T'(t) < 0$).

Example 1.1.5 (Torricelli's law). In fluid dynamics, *Torricelli's law* states that the instantaneous rate of change of the volume of a liquid inside a draining tank is proportional to the square root of the depth of the liquid. Let us model Torricelli's law by means of an ODE, assuming for simplicity that the draining tank has cylindrical shape with cross-sectional area $A > 0$. Let $V(t)$ and $y(t)$ denote, respectively, the volume and the depth of the liquid at time t. Then the law asserts that

$$V'(t) = -k\sqrt{y(t)} \tag{1.1.3}$$

[1] The reason for the terminology is obvious: the given family of functions is described by a single real parameter C.

[2] By way of contrast, usual algebraic equations such as polynomial equations in one variable have at most *finitely many* solutions.

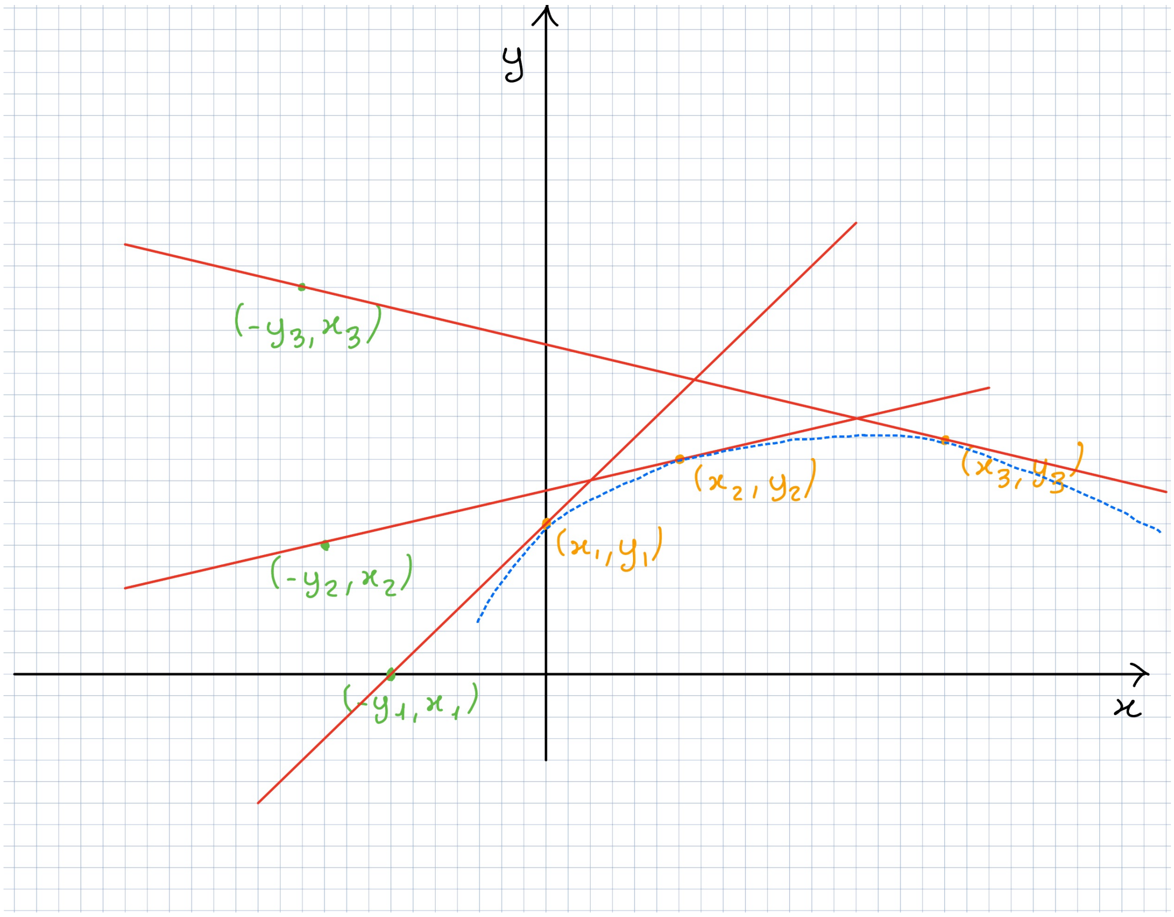

FIGURE 1.1. Here the red lines are tangent lines to the graph of g, and the dashed blue line represents a potential guess for the graph itself.

for some positive constant $k > 0$ (the amount of water in the tank is decreasing, thus $V'(t)$ must be negative).

At first sight, the differential relation (1.1.3) doesn't look like the differential equations we have encountered so far, in that there appear to be two unknown functions, namely $V(t)$ and $y(t)$. However, since the shape of the tank is cylindrical, there is a clear additional relation between the volume and the depth of the liquid, which is $V(t) = Ay(t)$. Since A is constant in time, (1.1.3) translates into

$$Ay'(t) = -k\sqrt{y(t)}\ ,$$

which is now a differential equation of the single unknown function $y(t)$.

We now present an example where a differential equation models a problem of geometric nature.

EXAMPLE 1.1.6. Let $g(x)$ be a real-valued function of a real variable. Suppose g satisfies the following geometric condition: for every point (x, y) in the graph[3] of g, the tangent line to the graph of g at (x, y) passes through the point $(-y, x)$. Figure 1.1 illustrates graphically the geometric requirement we are imposing on the graph. We shall see how to translate this geometric condition on the graph of g into a differential equation which is satisfied, that is, solved by g.

Fix a point (x, y) in the graph of g; this means that $y = g(x)$. We begin by finding the equation for the tangent line to the graph of g at the point $(x, g(x))$, after which we are going to impose that such line passes through $(-y, x) = (-g(x), x)$. By definition, the sought after

[3]Recall that the graph of a function $h(x)$ is the set of pairs $(x, h(x))$, which are pictorially identified with points in the xy-plane, where x varies over the domain of definition of h.

tangent line contains the point $(x, g(x))$ and has slope given by the derivative $g'(x)$ at the point x: its equation, using new variables s and t to avoid confusion, is thus

$$t(s) = g'(x)(s - x) + g(x) \ , \tag{1.1.4}$$

where we emphasize once again that x is fixed and s is the variable in the equation. If we impose now the condition that $(-g(x), x)$ lies on such tangent line, we obtain from (1.1.4) the relation

$$x = g'(x)(-g(x) - x) + g(x) \ ,$$

which is a *bona fide* ordinary differential equation, of the first order, satisfied by $g(x)$. Solving such an equation allows thus to determine all possible differentiable functions whose graph satisfies the geometric property phrased at the beginning.

1.1.3. A general framework for ordinary differential equations. We are now ready to give the formal definition of ordinary differential equation.

DEFINITION 1.1.7 (Ordinary differential equation). An **ordinary differential equation** (henceforth routinely abbreviated ODE) is an equation of the form

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \tag{1.1.5}$$

where $n \geq 1$ is an integer, F is a real-valued continuous function of $n + 2$ real variables, and $y(x)$ is the *unknown function* of the equation, which appears in it together with its derivatives $y'(x), y''(x), \ldots, y^{(n)}(x)$ and with the *independent variable* x.

The integer n is called the **order** of the ODE (1.1.5).

We shall say that (1.1.5) is an n-th order ODE; n corresponds to the highest order derivative appearing in the given ODE.

EXAMPLE 1.1.8. To digest the abstract definition, let us place the examples encountered so far within the general framework described by Definition 1.1.7.

(1) The equation

$$y'(x) = 2y(x) - x$$

which, upon renaming the unknown function and the independent variable, is precisely the one treated in Example 1.1.1, takes the form (1.1.5) for $n = 1$ and $F(t_1, t_2, t_3) = -t_1 + 2t_2 - t_3$, a real-valued function of three real variables t_1, t_2, t_3. Indeed, the equation $F(x, y(x), y'(x)) = 0$ amounts precisely to

$$0 = -x + 2y(x) - y'(x) \ , \quad \text{that is,} \quad y'(x) = 2y(x) - x \ .$$

Since the integer n is equal to 1 in this case, we have an example of a first-order differential equation.

(2) The equation

$$y''(x) + y'(x) - 3y(x) = \cos x \ ,$$

already discussed in Example 1.1.2, takes the form (1.1.5) for $n = 2$ and $F(t_1, t_2, t_3, t_4) = \cos t_1 + 3t_2 - t_3 - t_4$; to check this, simply plug the variables $(x, y(x), y'(x), y''(x))$ into (t_1, t_2, t_3, t_4), so as to obtain

$$0 = \cos x + 3y(x) - y'(x) - y''(x) \ , \quad \text{that is,} \quad y''(x) + y'(x) - 3y(x) = \cos x \ .$$

As $n = 2$, this is an example of a second-order ODE.

(3) Newton's law of cooling (Example 1.1.4) is expressed by the differential equation

$$y'(x) = -k(y(x) - A)$$

for given constants k and A. It is straightforward to verify, as in the two examples above, that we obtain the equation in the form (1.1.5) for $n = 1$ and $F(t_1, t_2, t_3) = k(t_2 - A) + t_3$; it is a first-order ODE.

(4) Torricelli's law (Example 1.1.5) is expressed by the differential equation

$$Ay'(x) = -k\sqrt{y(x)}$$

for given constants k and A. The equation takes the form (1.1.5) for $n = 1$ and $F(t_1, t_2, t_3) = k\sqrt{t_2} + At_3$; it is a first-order ODE.

We now formalize the rather intuitive notion of solution of an ODE. By an *open interval* in $\mathbb{R}$ we mean any set of real numbers of the form (a, b), thus with the boundary points a and b excluded, where a is a real number or $a = -\infty$ and $b > a$ is a real number, potentially $b = +\infty$.

Definition 1.1.9 (Solution of an ODE). A **solution of an ODE**

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

is a real-valued function $u(x)$ of a single real variable x, defined on some open interval $I \subset \mathbb{R}$, which is n-times continuously differentiable[4] on I and satisfies the equality

$$F(x, u(x), \dots, u^{(n)}(x)) = 0 \quad \text{for every } x \in I.$$

Example 1.1.10. Consider the function

$$u(x) = \frac{1}{C - x}\ , \tag{1.1.6}$$

where C is an arbitrary real constant. Since

$$u'(x) = \frac{1}{(C - x)^2}\ ,$$

we deduce that $u(x)$ is a solution to the differential equation

$$y'(x) = y^2(x)\ .$$

It is defined over two separate open intervals, namely $(-\infty, C)$ and $(C, +\infty)$. As C varies in $\mathbb{R}$, (1.1.6) describes a one-parameter family of solutions to the given first-order ODE.

Example 1.1.11. Let us verify that the function

$$u(x) = xe^{-x}$$

is a solution, defined over the whole real line, of the second-order ODE

$$y''(x) + 2y'(x) + y(x) = 0\ .$$

We compute first

$$u'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}$$

via the product rule for derivatives, and similarly

$$u''(x) = -e^{-x} - (1 - x)e^{-x} = e^{-x}(x - 2)\ .$$

Therefore, we obtain

$$u''(x) + 2u'(x) + u(x) = e^{-x}(x - 2) + e^{-x}(2 - 2x) + xe^{-x} = e^{-x}(x - 2 + 2 - 2x + x) = 0\ ,$$

which shows that $u(x) = xe^{-x}$ solves the given ODE.

Observe that the function

$$v(x) = e^{-x}$$

is also a solution: indeed, we have $v'(x) = -e^{-x}$ and $v''(x) = e^{-x}$, so that

$$v''(x) + 2v'(x) + v(x) = e^{-x} - 2e^{-x} + e^{-x} = 0\ ,$$

as desired.

[4] That is, it can be differentiated n times, and its n-th order derivative $u^{(n)}(x)$ is continuous.

In general, an ordinary differential equation may fail to admit any solution. For instance, the first-order ODE

$$y'^2(x) + y^2(x) = -1$$

does not admit any solution. Indeed, the square of any real number is nonnegative, whence

$$u'^2(x) + u^2(x) \geq 0$$

for any differentiable function $u(x)$.

It may also be the case that an ODE admits just one solution. This happens, for instance, of the second-order ODE

$$y''^2(x) + y^2(x) = 0 \ ,$$

which is only solved by the constant function $u = 0$.

The last two are, however, rather pathological examples; as we will amply discuss in the sequel, it is standard for an n-th order ODE to admit an *n-parameter family* of solutions, namely a collection of solutions which is described by n distinct real parameters.

Example 1.1.12. Let us go back to Example 1.1.11, i.e., to the second-order differential equation

$$y''(x) + 2y'(x) + y(x) = 0 \ . \tag{1.1.7}$$

We have verified that the two functions

$$u_1(x) = e^{-x} \ , \quad u_2(x) = xe^{-x}$$

are solutions to the equation. Linearity of derivatives allows us to deduce that any function of the form

$$u(x) = C_1 u_1(x) + C_2 u_2(x) \ , \tag{1.1.8}$$

where C_1 and C_2 are distinct real constants, is a solution: indeed, we compute

$$\begin{aligned} u''(x) + 2u'(x) + u(x) &= (C_1u_1(x) + C_2u_2(x))'' + 2(C_1u_1(x) + C_2u_2(x))' + C_1u_1(x) + C_2u_2(x) \\ &= C_1u_1''(x) + C_2u_2''(x) + 2(C_1u_1'(x) + C_2u_2'(x)) + C_1u_1(x) + C_2u_2(x) \ ; \end{aligned}$$

rearranging terms appropriately, we obtain

$$\begin{aligned} u''(x) + 2u'(x) + u(x) &= C_1(u_1''(x) + 2u_1'(x) + u_1(x)) + C_2(u_2''(x) + 2u_2'(x) + u_2(x)) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0 \ , \end{aligned}$$

as claimed.

We have thus found a two-parameter family of solutions to the second-order ODE (1.1.7); we shall develop solving strategies for such kind of equations which will enables us to ascertain that there are no other solutions, so that (1.1.8) completely describes the set of solutions to the given ODE.

1.1.4. Equilibrium solutions. One of the main goals of this course is to learn to analysize properties of solutions to differential equations. The most basic solutions to conceive are constant solutions.

Definition 1.1.13 (Equilibrium solution). A solution $u(x)$ of an ordinary differential equation

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \ ,$$

defined over a certain open interval $I \subset \mathbb{R}$, is called an **equilibrium solution**, or simply an **equilibrium**, if there is a real number C such that $u(x) = C$ for all $x \in I$.

It shall be important, whenever we attempt to study an ODE, to first single out its equilibrium solutions, if there are any. This will often be a basic step before implementing appropriate methods to find all other solutions.

Suppose the constant function $u(x) = C$ is a solution to the ODE

$$F(x, y(x), \ldots, y^{(n)}(x)) = 0 \ ;$$

by definition, this means that

$$0 = F(x, u(x), u'(x), \ldots, u^{(n)}(x)) = F(x, C, 0, \ldots, 0) ,$$

the last equality holding since all derivatives of a constant function vanish identically. Therefore, the real number C is a solution to the functional equation

$$F(x, C, 0, \ldots, 0) = 0 ,$$

meaning that $F(x, C, 0, \ldots, 0) = 0$ for all x in the domain of definition of F. Conversely, it is clear that if $C \in \mathbb{R}$ solves the last displayed equation, then the constant function $u(x) = C$ is a solution to the given ODE.

Example 1.1.14. Let's determine all equilibrium solutions of the first-order ODE

$$y'(x) = (M - y(x))(y^2(x) - 3y(x) + 2) ,$$

where M is a given real number. A constant function $u(x) = C$ solves the equation if and only if

$$0 = u'(x) = (M - C)(C^2 - 3C + 2) = (M - C)(C - 1)(C - 2) ,$$

which is an algebraic equation in the variable C with solutions $C = 1$, $C = 2$ and $C = M$. Therefore the equilibrium solutions of the ODE at hand are

$$u(x) = 1 , \quad u(x) = 2 , \quad u(x) = M .$$

Example 1.1.15. Consider the second-order ODE

$$y''(x) - y'(x) + 3ky(x) = 0 ,$$

where $k \in \mathbb{R}$ is given. Let's determine the equilibrium solutions: a constant function $u(x) = C$ solves the ODE if and only if

$$0 = u''(x) - u'(x) + 3ku(x) = 0 + 0 + 3kC = 3kC .$$

We have thus two distinct regimes according to the value of k: if $k \neq 0$, then the last displayed algebraic equation is only solved for $C = 0$, which produces the unique equilibrium solution

$$u(x) = 0 .$$

On the other hand, if $k = 0$, then the equation $0 = 3kC$ is always verified, no matter the value of C; in this case, we thus have a one-parameter family of equilibrium solutions to the given ODE,

$$u(x) = C , \quad C \in \mathbb{R}.$$

Example 1.1.16. Consider the first-order ODE

$$y'(x) = y(x) \cos x - e^x .$$

Let $u(x) = C$ be a constant function: can it be a solution to the given equation? For it to be the case, we must have

$$0 = u'(x) = u(x) \cos x - e^x = C \cos x - e^x ,$$

for all real values of x. It is clear that there exists no real number C for which this is verified, since $C \cos x$ is a bounded function, whereas e^x is unbounded. Thus, the given ODE has no equilibrium solutions.

1.1.5. Initial value problems. In applications, differential equations customarily appear in conjunction with *initial conditions*: in the study of the time-evolution a certain physical quantity $y(t)$, we typically know its value y_0 at a given moment in time t_0, and we understand the physical law underlying its evolution, expressed by a differential equation for $y(t)$. Assuming such knowledge, we would like to determine the future evolution of $y(t)$ completely, namely all the values $y(t)$ for all $t > t_0$. A problem of such nature is known as *initial value problem*.

DEFINITION 1.1.17 (Initial value problem). An **initial value problem** (IVP in abridged form) is a pair

$$\begin{cases} F(x, y(x), \ldots, y^{(n)}(x)) = 0 \\ y(x_0) = y_0 \ , \ y'(x_0) = y_1 \ , \ \ldots \ , \ y^{(n-1)}(x_0) = y_{n-1} \end{cases} \tag{1.1.9}$$

consisting of an ordinary differential equation

$$F(x, y(x), \ldots, y^{(n)}(x)) = 0 \tag{1.1.10}$$

and a set of **initial conditions**

$$y(x_0) = y_0 \ , \ y'(x_0) = y_1 \ , \ \ldots \ , \ y^{(n-1)}(x_0) = y_{n-1}$$

where $x_0, y_0, \ldots, y_{n-1}$ are real numbers.

A **solution of the IVP** (1.1.9) is a solution $u(x)$ of the ODE (1.1.10) which is defined on an interval I containing the point x_0, and which satisfies the conditions

$$u(x_0) = y_0 \ , \ u'(x_0) = y_1 \ , \ \ldots \ , \ u^{(n-1)}(x_0) = y_{n-1} \ .$$

EXAMPLE 1.1.18. Let us find a solution to the IVP

$$\begin{cases} y'(x) = y^2(x) \\ y(1) = 1 \end{cases}$$

taking advantage of the family of solutions found in Example 1.1.10. From the latter, we know that each function

$$u(x) = \frac{1}{C - x} \ ,$$

for $C \in R$, is a solution to the ODE in the given initial value problem. We now *impose the initial condition* $u(1) = 1$ prescribed by the IVP, and obtain the algebraic equation

$$1 = u(1) = \frac{1}{C - 1} \ ,$$

from which we readily get $C = 2$. Therefore, the function

$$u(x) = \frac{1}{2 - x}$$

is a solution to the given IVP.

It is now natural to ask: are there any more solutions? Certainly none of the form $1/(C-x)$ for $C \neq 2$, since we obtained $C = 2$ precisely by dictating the initial condition. In principle, there might however be other solutions to the ODE $y'(x) = y^2(x)$ which are not of the form $u(x) = 1/(C - x)$. In later sections of this chapter we will *prove* that there are, as a matter of fact, no other solutions.

EXAMPLE 1.1.19. Leveraging the family of solutions found in Example 1.1.12, let us find a solution to the IVP

$$\begin{cases} y''(x) + 2y'(x) + y(x) = 0 \\ y(0) = 0 \ , \ y'(0) = -1 \end{cases} .$$

A *general solution* to the given DE is, as we already verified,

$$u(x) = C_1 e^{-x} + C_2 x e^{-x} \tag{1.1.11}$$

for real parameters C_1, C_2. We now impose the two initial conditions:

$$0 = u(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1 \ ,$$

so that we find $C_1 = 0$ and $u(x) = C_2 x e^{-x}$ for some $C_2 \in \mathbb{R}$, which we determine imposing the second initial condition. We compute $u'(x) = C_2(e^{-x} - xe^{-x})$, and thus

$$-1 = u'(0) = C_2(1 - 0) = C_2 \ .$$

Therefore, the unique function $u(x)$ in the family (1.1.11) which solves the original IVP is

$$u(x) = -xe^{-x} \ .$$

A major achievement of the general mathematical theory of ordinary differential equations is that, under rather mild assumptions, initial value problems *always admit a unique solution* defined for all values of the independent variable x which are sufficiently close to the initial value x_0. Thus, while a n-th order ODE usually admits an n-parameter family of solutions, the additional datum of n initial conditions in an IVP forces uniqueness. In this course, we shall not be concerned with the abstract theory of ODEs, and will rather verify the aforementioned uniqueness principle in a wide variety of specific examples.

1.1.6. Ordinary differential equations in normal form. Throughout this course, we shall exclusively deal with ordinary differential equations expressed in **normal form**, namely those expressing the highest-order derivative of the unknown function as a function of all the remaining derivatives: more precisely, an n-th order ODE in normal form appears as

$$y^{(n)}(x) = F(x, y(x), \ldots, y^{(n-1)}(x))$$

for a certain continuous real-valued function F of $n + 1$ real variables.

EXAMPLE 1.1.20. The first-order ODE

$$y'(x) = 2x \log y(x)$$

is in normal form, whereas the first-order ODE

$$y'^2(x) + y^2(x) = x^4$$

is not in normal form. Notice that trying to solve the latter for the highest-order derivative $y'(x)$ would produce the ambiguity

$$y'(x) = \pm\sqrt{x^4 - y^2(x)}$$

in the choice of square root.

EXAMPLE 1.1.21. The second-order ODE

$$t^3 x''(t) - t^2 x(t) = t - \sin t$$

is not in normal form, while the second-order ODE

$$x''(t) = -tx'(t) - x(t) + 1$$

is in normal form.

For the sake of brevity, we adopt the following terminological convention.

CONVENTION. From now on, unless explicitly mentioned, a differential equation without further specification is meant to be an ordinary differential equation, and will routinely be abbreviated as DE.

1.2. Integrals as general and particular solutions

We now begin a systematic study of first-order differential equations in normal form, that is, equations of the form

$$y'(x) = f(x, y(x))$$

in the unknown $y(x)$, where f is a (given) function of two real-variables. This section is devoted to the analysis of the most elementary instances of such equations, namely the case where the function f only depends on the independent variable x. The resulting form of the equation is thus

$$y'(x) = f(x)\ . \tag{1.2.1}$$

Direct integration yields all solutions to the last-displayed equation. Indeed, here we have a fixed, known continuous function $f(x)$, and we look for all continuously differentiable functions $y(x)$ whose derivative is given by the function f. According to the terminology introduced in Calculus 2, $y(x)$ solves the DE in (1.2.1) if and only if $y(x)$ is an *anti-derivative* of the function $f(x)$. Anti-derivatives are given by *indefinite integrals*, whence $y(x)$ is a solution if and only if

$$y(x) = \int f(x)\ \mathrm{d}x = g(x) + C \tag{1.2.2}$$

where $g(x)$ is a choice of an anti-derivative of $f(x)$, and C is a real constant. What we just described in (1.2.2) is routinely referred to as **a general solution** of the DE in (1.2.1), namely a collection of solutions parametrized, in this case, by the constant C. For each fixed $C \in \mathbb{R}$, we obtain a **particular solution** of the equation in (1.2.1); thus a general solution is a family of particular solutions. In this case, we are dealing with a one-parameter family of solutions, as shall customarily be the case for first-order differential equations.

If a general solution to a given equation comprises *all* possible solutions, then we shall speak of **the general solution** of the equation. In the present case, (1.2.2) provides the general solution to (1.2.1); indeed, if $g(x)$ is a fixed anti-derivative of $f(x)$ and $u(x)$ is any solution to (1.2.1), namely satisfies $u'(x) = f(x)$, then a well known theorem of calculus tells us that u and g must differ by a constant, for they have the same derivative. Hence, $u(x) = g(x) + C$ for some $C \in \mathbb{R}$ and thus u belongs to the family of functions described in (1.2.2).

Recall that an anti-derivative $g(x)$ of $f(x)$ is given by any *definite integral* of the form

$$g(x) = \int_{x_0}^{x} f(t)\ \mathrm{d}t$$

where x_0 is a real number in a given open interval on which f is defined; this is indeed the content of the fundamental theorem of calculus.

Initial conditions enable to specialize a general solution to a particular solution. Suppose given an IVP

$$\begin{cases} y'(x) = f(x) \\ y(x_0) = y_0 \end{cases} \tag{1.2.3}$$

where the differential equation is of the kind we are studying in this section. We known from the discussion above that a solution to the DE in (1.2.3) must take the form

$$y(x) = g(x) + C$$

where g is a fixed anti-derivative of f and C is a real number. Imposing the initial condition $y(x_0) = y_0$ yields

$$y_0 = y(x_0) = g(x_0) + C\ ,$$

from which we derive

$$C = y_0 - g(x_0)\ .$$

Therefore, we have shown that the IVP in (1.2.3) admits a *unique solution*, which is given by

$$y(x) = g(x) + y_0 - g(x_0)$$

for any fixed anti-derivative $g(x)$ of $f(x)$. If, for instance, we choose $g(x)$ to be the definite integral

$$g(x) = \int_{x_0}^{x} f(t)\,\mathrm{d}t\ ,$$

then $g(x_0) = 0$ and thus the unique solution can be expressed as

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,\mathrm{d}t\ .$$

We summarize the results obtained so far in this section in the following theorem.

THEOREM 1.2.1. *Let $f(x)$ be a continuous function defined on an open interval $I = (a, b) \subset \mathbb{R}$, and consider the first-order differential equation*

$$y'(x) = f(x)\ .$$

Let $g(x)$ be an anti-derivative of $f(x)$ on I. Then, a continuously differentiable function $u(x)$, defined on I, is a solution of the given differential equation if and only if

$$u(x) = g(x) + C$$

for some $C \in \mathbb{R}$.

Furthermore, if $x_0 \in I$ and $y_0 \in \mathbb{R}$, the initial value problem

$$\begin{cases} y'(x) = f(x) \\ y(x_0) = y_0 \end{cases}$$

admits a unique solution $u(x)$ defined on I, which is given by

$$u(x) = y_0 + \int_{x_0}^{x} f(t)\,\mathrm{d}t\ .$$

We now familiarize ourselves with the method by working out a few examples.

EXAMPLE 1.2.2. Consider the IVP

$$\begin{cases} y'(x) = x + 4 \\ y(1) = 3 \end{cases} .$$

We first find the general solution to

$$y'(x) = x + 4$$

by means of indefinite integrals:

$$y(x) = \int x + 4\,\mathrm{d}x = \frac{x^2}{2} + 4x + C\ , \quad C \in \mathbb{R}.$$

Now, we impose the condition

$$y(1) = 3$$

to find the appropriate value of C; we have

$$3 = y(1) = \frac{1}{2} + 4 + C\ ,$$

from which

$$C = 3 - \frac{1}{2} - 4 = -\frac{3}{2}\ .$$

We conclude that the unique solution to the given IVP is the function

$$y(x) = \frac{x^2}{2} + 4x - \frac{3}{2}\ .$$

Example 1.2.3. Consider the IVP

$$\begin{cases} y'(x) = \frac{1}{\sqrt{x+1}} \\ y(0) = 2 \end{cases} .$$

The function $f(x) = \frac{1}{\sqrt{x+1}}$ is defined over the open interval $\{x : x+1 > 0\} = (-1, +\infty)$. The general solution to

$$y'(x) = \frac{1}{\sqrt{x+1}}$$

is given by

$$y(x) = \int \frac{1}{\sqrt{x+1}}\,\mathrm{d}x = 2\sqrt{x+1} + C\ , \quad C \in \mathbb{R}.$$

Plugging the initial condition

$$y(0) = 2$$

yields

$$2 = y(0) = 2 + C\ ,$$

which gives $C = 0$. Thus the unique solution to the given IVP is the function

$$y(x) = 2\sqrt{x+1}\ .$$

Example 1.2.4. Let's find the general solution to the first-order DE

$$y'(x) = e^x \cos x\ .$$

It is given by the indefinite integral

$$y(x) = \int e^x \cos x\,\mathrm{d}x\ ,$$

which we handle through integration by parts (see Appendix A.1): setting $u = e^x$ and $v' = \cos x$, we get

$$\int e^x \cos x\,\mathrm{d}x = e^x \sin x - \int e^x \sin x\,\mathrm{d}x\ .$$

We now apply again integration by parts to the last displayed integral, with $u = e^x$ and $v' = \sin x$:

$$\int e^x \sin x\,\mathrm{d}x = -e^x \cos x + \int e^x \cos x\,\mathrm{d}x\ .$$

Combining the two last displayed equations, we obtain

$$\int e^x \cos x\,\mathrm{d}x = e^x \sin x + e^x \cos x - \int e^x \cos x\,\mathrm{d}x\ ,$$

from which it follows that

$$\int e^x \cos x\,\mathrm{d}x = \frac{e^x}{2}(\cos x + \sin x) + C\ , \quad C \in \mathbb{R}.$$

The general solution to the DE $y'(x) = e^x \cos x$ is thus given by

$$y(x) = \frac{e^x}{2}(\cos x + \sin x) + C\ , \quad C \in \mathbb{R}.$$

Example 1.2.5. We study the IVP

$$\begin{cases} y'(x) = \frac{21}{4+x^2} \\ y(0) = \pi \end{cases} .$$

The general solution to

$$y'(x) = \frac{21}{4+x^2}$$

is given by

$$y(x) = \int \frac{21}{4+x^2}\,\mathrm{d}x = \frac{21}{4}\int \frac{1}{1+(x/2)^2}\,\mathrm{d}x = \frac{21}{2}\arctan(x/2) + C\ , \quad C \in \mathbb{R}.$$

Imposing the initial condition

$$y(0) = \pi$$

yields

$$\pi = \frac{21}{2}\arctan(0) + C = 0 + C = C\ ,$$

whence the unique solution to the given IVP is

$$y(x) = \frac{21}{2}\arctan(x/2) + \pi\ .$$

1.2.1. Position, velocity and acceleration. The analysis conducted so far in this section allows to fully understand the motion of a *point mass* confined to stay on a line and subject to external forces which only depend on the time variable, and not on other physical data such as the position or the velocity of the object.

Recall that the position of the object, which is described by a single real number in view of the dimensional constraint on the motion, can be recorded via a function $x(t)$ of the time variable t. The *velocity* is then given by the first derivative $v(t) = x'(t)$, and the *acceleration* by the first derivative of the velocity $a(t) = v'(t)$, or equivalently the second derivative of the position $a(t) = x''(t)$.

Newton's second law of motion asserts that the acceleration which a given object undergoes is directly proportional to the overall force imparted on it, via a proportionality constant which is the inverse of the mass m. Its most concise (and best known) formulation is

$$F = ma\ . \tag{1.2.4}$$

Delving deeper into the nature of such equation, we find out that it is a second-order ordinary differential equation; indeed, the force F is, in many relevant instances, a function of the time t, of the position $x(t)$ and of the velocity $v(t)$. Thus (1.2.4) amounts to

$$mx''(t) = F(t, x(t), x'(t)) \tag{1.2.5}$$

for a force-function $F(t, x(t), x'(t))$ which is a given datum of the problem. Now (1.2.5) is precisely a second-order ODE expressed in normal form.

Given now initial conditions $x(t_0) = x_0$, $v(t_0) = v_0$ on the position and velocity, the principle of *Newton's determinism* affirms that the evolution of the motion of the object is uniquely prescribed for all future (and past) times $t > t_0$. Mathematically, this is a consequence of a general theorem in the theory of ordinary differential equations, which establishes existence and uniqueness of the solution to an IVP of the form

$$\begin{cases} mx''(t) = F(t, x(t), x'(t)) \\ x(t_0) = x_0\ ,\ x'(t_0) = v_0 \end{cases}$$

for all functions F satisfying a rather mild regularity condition.

Whilst we are not concerned with a theorem of such general nature in this course, let us verify this existence and uniqueness principle in action, under the assumption that the force depends only on the time variable, which places ourselves within the framwork of the present section. We are thus faced with the IVP

$$\begin{cases} mx''(t) = F(t) \\ x(t_0) = x_0\ ,\ x'(t_0) = v_0 \end{cases}$$

for a given function $F(t)$ of one real variable. The first observation is that we can rewrite the last displayed second-order IVP as two distinct first-order IVPs, one for the unknown $v(t)$

and the other for the unknown $x(t)$. Indeed, the previous is equivalent to the two initial value problems

$$\begin{cases} mv'(t) = F(t) \\ v(t_0) = v_0 \end{cases} , \qquad \begin{cases} x'(t) = v(t) \\ x(t_0) = x_0 \end{cases} . \tag{1.2.6}$$

We begin by solving the first one, via the method discussed at the beginning of the present section. Writing

$$v'(t) = \frac{1}{m} F(t)$$

for the differential equation, we integrate to obtain the general solution

$$v(t) = \frac{1}{m} \int F(t) \, \mathrm{d}t = g(t) + C \ , \quad C \in \mathbb{R},$$

where we may take for instance

$$g(t) = \int_{t_0}^{t} F(s) \, \mathrm{d}s$$

as anti-derivative of F. Imposing the initial condition $v(t_0) = v_0$ yields

$$v_0 = \int_{t_0}^{t_0} F(s) \, \mathrm{d}s + C = 0 + C = C \ ,$$

whence the unique solution to the first IVP in (1.2.6) is given by

$$v(t) = v_0 + \int_{t_0}^{t} F(s) \, \mathrm{d}s \ .$$

Now $v(t)$ becomes a known function of t, and our final goal is to solve the second IVP in (1.2.6) with this given datum. Arguing as above, we deduce that the unique solution the second IVP is given by

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} v(s) \, \mathrm{d}s = x_0 + \int_{t_0}^{t} v_0 \, \mathrm{d}s + \int_{t_0}^{t} \int_{t_0}^{s} F(r) \, \mathrm{d}r \\ &= x_0 + v_0(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{s} F(r) \, \mathrm{d}r \ , \end{aligned} \tag{1.2.7}$$

which completely determines the position function $x(t)$ for all times t in terms of the initial conditions x_0, v_0 and the force function $F(t)$.

Suppose, for instance, that the force F is constant in time: $F(t) = F_0$ for a given real number F_0. Then (1.2.7) boils down to

$$x(t) = x_0 + v_0(t - t_0) + \int_{t_0}^{t} F_0(s - t_0) \, \mathrm{d}s = x_0 + v_0(t - t_0) + \frac{F_0}{2}(t - t_0)^2 \ . \tag{1.2.8}$$

Therefore, we infer that, in the presence of a constant force, the position function is quadratic in time. On the other hand, the velocity function

$$v(t) = x + (t) = v_0 + F_0(t - t_0) \tag{1.2.9}$$

is linear in time.

Remark 1.2.6. We recover here the familiar principle, known as *Newton's first law of motion*, that an object under the influence of no external forces persists in his constant-velocity motion, with a position function thus changing proportionally with time. Indeed, if $F_0 = 0$, we find from (1.2.9) that

$$v(t) = v_0$$

and from (1.2.8) that

$$x(t) = x_0 + v_0(t - t_0) \ .$$

Let us now determine the motion of an object in some concrete cases, starting from knowledge of the initial position and velocity, as well as of the force to which it is subject.

Example 1.2.7 (Vertical motion with gravitational acceleration). Suppose our given point mass is only subject to the gravitational attraction exerted by the Earth. The resulting force is proportional, via the object mass, to the gravitational acceleration g ($g \sim 9.8\ m/s^2$), which can approximately be considered constant in the promixity of the ground: denoting the position function of the point mass by $y(t)$[5], and assuming the y-axis is oriented upwards, we can thus write

$$F = -mg \ ,$$

since $g > 0$ and the force points downwards. For simplicity, and without loss of generality, let us assume we know initial position and velocity, y_0 and v_0 respectively, at time $t = 0$. We first find the general solution for the velocity function, which satisfies

$$v'(t) = a(t) = \frac{1}{m}F(t) = -\frac{1}{m}mg = -g \ ,$$

and is therefore given by

$$v(t) = \int a(t)\ \mathrm{d}t = \int -g\ \mathrm{d}t = -gt + C \ , \quad C \in \mathbb{R}.$$

Plugging the initial condition $v(0) = v_0$ we deduce that

$$v_0 = -g \cdot 0 + C = C \ ,$$

whence the velocity function is given by

$$v(t) = v_0 - gt \ .$$

We proceed to find the position function, which satisfies the IVP

$$\begin{cases} y'(t) = v(t) \\ y(0) = y_0 \end{cases} \quad ;$$

as general solution we obtain, by integration,

$$y(t) = \int v(t)\ \mathrm{d}t = \int v_0 - gt\ \mathrm{d}t = v_0 t - \frac{1}{2}gt^2 + C \ , \quad C \in \mathbb{R}.$$

Inserting the initial condition $y(0) = y_0$ results into

$$y_0 = v_0 \cdot 0 - \frac{g}{2} \cdot 0^2 + C = C \ ,$$

so that the position function is given by

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \ .$$

If we were to draw the graph of the position as a function of time, we would thus find a parabola[6] intercepting the y-axis at the initial position y_0.

Example 1.2.8. Suppose an arrow is shot straight upward from the ground with initial velocity $v_0 = 20m/s$. What is the highest point it reaches in the air?

Observe first that, from the previous example, the position function of the arrow is given by (notice that $y_0 = 0$ since the arrow is shot from the ground)

$$y(t) = -\frac{1}{2}9.8t^2 + 20t \ .$$

[5] The notation points to the fact that that the motion is vertical.

[6] Notice that this has nothing to do with the familiar real-world experience of observing a parabola-like trajectory when throwing an object in the air with a certain non-zero angle with respect to the vertical direction. Here we are dealing with a motion which is assumed to occur only in the vertical direction.

The highest point reached by the arrow corresponds the moment where the velocity vanishes; since the latter is given by

$$v(t) = y'(t) = -9.8t + 20 ,$$

the time instant t_0 where this occurs must verify $0 = -9.8t_0 + 20$, that is,

$$t_0 = 20/9.8 \sim 2s .$$

The position of the arrow at that moment is given by

$$y(t_0) = -\frac{1}{2}9.8t_0^2 + 20t_0 \sim -2 \cdot 9.8 + 40 \sim 20m .$$

Let us expand upon the last example by treating the general case. Suppose we want to find the highest point reached by an object whose position function is given by

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 ,$$

where it is physically reasonable to assume that $y_0, v_0 \geq 0$. The velocity function

$$v(t) = y'(t) = -gt + v_0$$

vanishes if and only if

$$t = v_0/g ,$$

which corresponds to the position

$$y_{\text{highest}} = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g} + y_0 = y_0 + \frac{v_0^2}{g} ,$$

which is consistent with the intuition that, the higher the initial position and the initial velocity are, the higher is the top of the trajectory followed by the object, with a dependency which is linear in the initial position and quadratic in the initial velocity.

APPENDIX A

Integration techniques

A.1. Integration by parts