Spatial and temporal central limit theorems for flows on negatively curved hyperbolic surfaces

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Free motion and randomness

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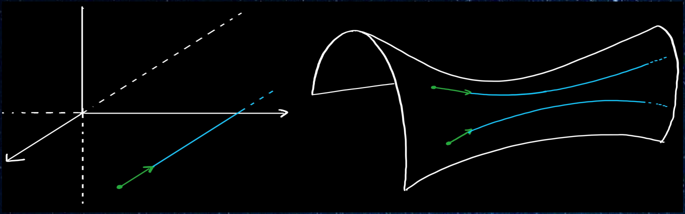

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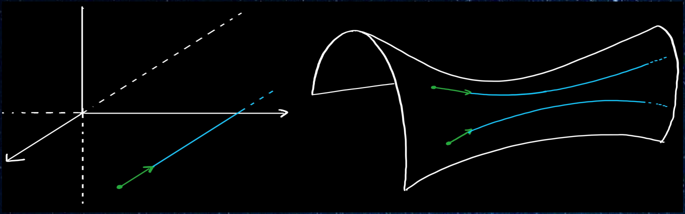
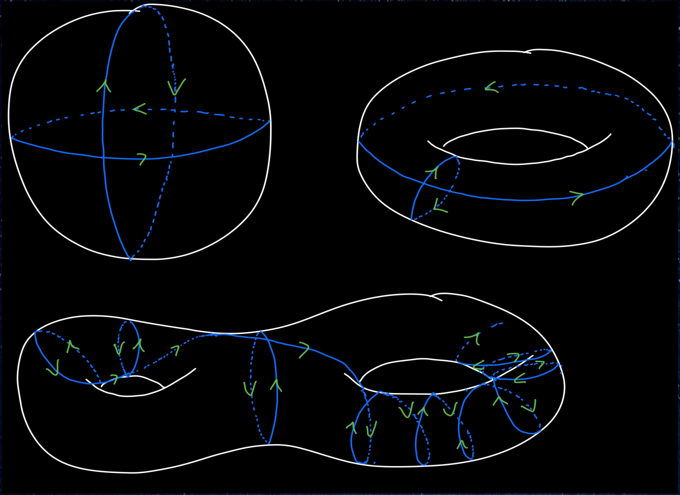


To what extent is the qualitative behaviour of its trajectory predictable?

At times negativity is uplifting...

The answer depends heavily on the curvature properties of the manifold.



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Example: any compact, connected smooth embedded surface $S \subset \mathbb{R}^3$, g_p being the restriction to $T_pS \leq T_p\mathbb{R}^3 \simeq \mathbb{R}^3$ of the standard inner product.

A mathematical framework: the geodesic flow

Define the tangent bundle

$$TS = \bigsqcup_{p \in S} T_p S = \{(p, v) : p \in S, v \in T_p S\}$$

and the unit tangent bundle $T^1 S = \{(p, v) \in TS : \|v\|_g = 1\}$.

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Hopf-Rinow's theorem: for any $(p, v) \in T^1S$ there is a (unique) smooth curve $\gamma_{p,v} : \mathbb{R} \to S$ such that

- γ minimizes distances locally: for every $t_0 \in \mathbb{R}$ there is $\delta > 0$ such that $L(\gamma|_{[t_0-\delta, t_0+\delta]}) = d_g(\gamma(t_0 - \delta), \gamma(t_0 + \delta))$;
- $\gamma(0) = p$, $\gamma'(0) = v$;
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We may thus define, for any $t \in \mathbb{R}$,

$$\begin{aligned} g_t \colon T^1S &\longrightarrow T^1S \\ (p, v) &\mapsto (\gamma_{p,v}(t), \gamma'_{p,v}(t)) \end{aligned} \quad .$$

A zoo of geodesic orbits

The collection $(g_t)_{t\in\mathbb{R}}$ defines a smooth flow, called the **geodesic flow**, on T^1S: the map $\mathbb{R}\times T^1S\to T^1S$, $(t,(p,v))\mapsto g_t(p,v)$ is smooth and $g_t\circ g_s=g_{t+s}$ for any $t,s\in\mathbb{R}$. Geodesics on the surface S are projections of geodesic orbits $(g_t(p,v))_{t\in\mathbb{R}}$.

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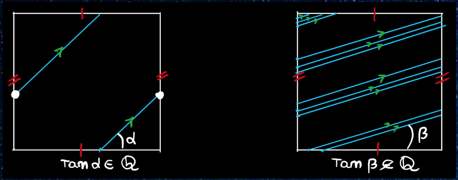

They might be compact, dense or simply wild.

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Standing assumption: S is compact, and $\mathrm{vol}_S(S) = 1$.

The geodesic flow as a stationary process

The *Liouville measure* m_{T^1S} on T^1S is defined weakly via

$$\int_{T^1S} f \, \mathrm{d}m_{T^1S} := \int_S \int_{T_p^1 S} f(p,v) \, \mathrm{d}\theta_p(v) \, \mathrm{d}\,\mathrm{vol}_S(p) \text{ for any } f \in \mathrm{C}(T^1S),$$

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Liouville's theorem (1838): m_{T^1S} is invariant under $(g_t)_{t\in\mathbb{R}}$, that is

$$m_{T^1S}(g_t(B)) = m_{T^1S}(B) \text{ for any } t \in \mathbb{R} \text{ and any measurable } B \subset T^1S.$$

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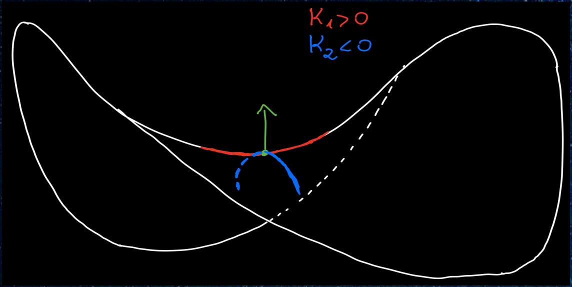
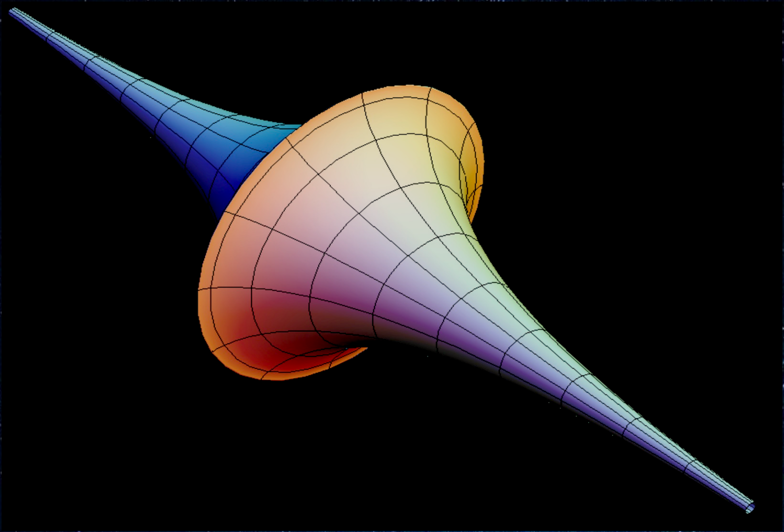
Overarching paradigm: in negative curvature, the $f \circ g_t$ behave much like independent random variables.

The Gaussian curvature

Suppose $S \subset \mathbb{R}^3$. Fix a point $p \in S$ and a unit vector $\overrightarrow{n}$ attached to p and normal to S. For any plane $\Pi \ni \overrightarrow{n}$, the intersection $\Pi \cap S$ is a smooth curve inside Π, having a well-defined signed curvature κ with respect to its normal vector $\overrightarrow{n}$. The *principal curvatures* κ_1, κ_2 of S at p are the supremum and the infimum of all the curvatures obtained by varying Π. The *Gaussian curvature* of S at p is the product $K = \kappa_1 \kappa_2$.



A negatively curved surface: the pseudosphere



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A first embodiment of randomness is ergodicity. If S has Gaussian curvature $K < 0$, the geodesic flow $(g_t)_{t\in\mathbb{R}}$ is *ergodic* with respect to the Liouville measure m_{T^1S}:

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This yields at once remarkable chaoticity features of geodesics: m_{T^1S}-almost surely, (forward) geodesic orbits *equidistribute* in the ambient space: for any open $V \subset T^1S$,

$$\frac{1}{T} m_{\mathbb{R}}\{0 \leq t \leq T : g_t x \in V\} \overset{T\to+\infty}{\longrightarrow} m_{T^1S}(V) \ .$$

A strong law of large numbers

Theorem (Birkhoff's pointwise ergodic theorem)

If $(\phi_t)_{t\in\mathbb{R}}$ *is a measure-preserving ergodic flow on a probability space* $(X, \mathcal{B}, \mu)$ *and* $f : X \to \mathbb{R}$ *is a* μ*-integrable function, then*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\phi_t x)\, dt = \int_X f\, d\mu$$

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Equivalently: if $(X_t)_{t\geq 0}$ is a stationary ergodic E-valued process, then a SLLN holds for any statistic $f : E \to \mathbb{R}$ with finite expectation:

$$\frac{1}{T} \int_0^T f(X_t)\, \mathrm{d}t \overset{T\to\infty}{\longrightarrow} \mathbb{E}[f(X_0)] \quad \mathbb{P}\text{-almost surely.}$$

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In general, the CLT fails for ergodic systems.

Consider $X = \{0, 1\}$, $\phi \colon X \to X$ defined by $\phi(0) = 1$, $\phi(1) = 0$, and $\mu = \frac{1}{2}(\delta_0 + \delta_1)$; then ϕ is ergodic with respect to μ. However, for any $f \colon X \to \mathbb{R}$ with zero mean, the random variables $\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f \circ \phi^n$ do <u>not</u> converge in distribution towards a Gaussian random variable.

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On the other hand, the abundance of possible correlation properties for processes of the form $(f \circ \phi_t)_t$ accounts for the emergence of various distributional behaviours in the limit.

Therefore, we need a general framework to study the limiting distributions of *ergodic integrals* $I_T(f, x) = \cancel{\frac{1}{T}} \int_0^T f(\phi_t x)\, \mathrm{d}t$.

Spatial distributional limit theorem

Definition (Spatial DLT)

Let $(X, \mathcal{B}, \mu, (\phi_t)_{t\in\mathbb{R}})$ be a measure-preserving flow. The ergodic integrals of $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$ satisfy a *spatial distributional limit theorem* if there exist real functions $(A_T), (B_T)$, with $B_T \to +\infty$ as $T \to +\infty$, and a non-trivial real-valued random variable Y such that the random variables $Y_T = \frac{I_T(f) - A_T}{B_T}$ converge in distribution towards Y as $T \to +\infty$.

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Theorem (Sinai, Ratner)

Suppose S has negative Gaussian curvature. The ergodic integrals, along the geodesic flow, of any smooth function $f\colon T^1S \to \mathbb{R}$ not cohomologous to a constant satisfy a spatial DLT. Specifically

$$\frac{I_T(f) - T\int_{T^1M} f\, dm_{T^1S}}{\sigma\sqrt{T}} \xrightarrow{T\to\infty} \mathcal{N}(0,1) \text{ in distribution, for some } \sigma > 0.$$

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$$\left(\frac{1}{\sqrt{n}} S_t^{(n)}\right)_{0\leq t\leq 1} \overset{n\to\infty}{\longrightarrow} (B_t)_{0\leq t\leq 1} \quad \text{in distribution,}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

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In this regard, the notion of Almost Sure Invariance Principle (ASIP) has been introduced by Strassen in the context of martingales, and later extended by Philipps and Stout to more general weakly dependent processes. It formalizes the intuition that trajectories of certain random processes are well approximable by Brownian trajectories.

The Almost Sure Invariance Principle

In the setting of ergodic integrals, it may be phrased as follows:

Definition (Almost Sure Invariance Principle)

Let $(X, \mathcal{B}, \mu, (\phi_t)_{t\in\mathbb{R}})$ be a measure-preserving flow. The ergodic integrals $I_t(f)$ of $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$ satisfy the **Almost Sure Invariance Principle** if there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, two processes $(I'_t)_{t\geq 0}, (B_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\sigma > 0$ such that:

1. $(I'_t)_{t\geq 0}$ has the same law as $(I_t(f))_{t\geq 0}$;
2. $(B_t)_{t\geq 0}$ is a standard Brownian motion;
3. for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$|I'_t(\omega) - B_{\sigma^2 t}(\omega)| = o(t^{1/2}).$$

The Almost Sure Invariance Principle

In the setting of ergodic integrals, it may be phrased as follows:

Definition (Almost Sure Invariance Principle)

Let $(X, \mathcal{B}, \mu, (\phi_t)_{t\in\mathbb{R}})$ be a measure-preserving flow. The ergodic integrals $I_t(f)$ of $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$ satisfy the **Almost Sure Invariance Principle** if there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, two processes $(I'_t)_{t\geq 0}, (B_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\sigma > 0$ such that:

1. $(I'_t)_{t\geq 0}$ has the same law as $(I_t(f))_{t\geq 0}$;
2. $(B_t)_{t\geq 0}$ is a standard Brownian motion;
3. for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$|I'_t(\omega) - B_{\sigma^2 t}(\omega)| = o(t^{1/2}).$$

ASIP $\Longrightarrow$ Donsker's IP

Consequences of the ASIP

If the ergodic integrals of f fulfill the ASIP, then the processes $(I_t^{(n)}(f))_{t\geq 0}$ defined by $I_t^{(n)}(f) := I_{nt}(f)$ satisfy

$$\left(\frac{1}{\sigma\sqrt{n}} I_t^{(n)}(f)\right)_{0\leq t\leq 1} \overset{n\to\infty}{\longrightarrow} (B_t)_{0\leq t\leq 1} \quad \text{in distribution,}$$

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As mentioned earlier, we have:

Theorem (Denker, Philipp)

If S has negative Gaussian curvature and $f\colon T^1S \to \mathbb{R}$ is a smooth function, not cohomologous to a constant, with $\int_{T^1S} f \, dm_{T^1S} = 0$, then the ergodic integrals of f along $(g_t)_{t\in\mathbb{R}}$ satisfy the ASIP.

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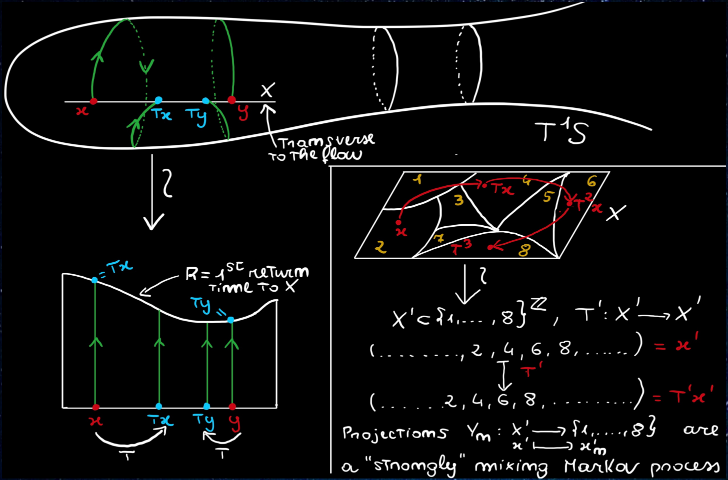
As is well-known, the ASIP enables to transfer typical features of Brownian trajectories to the process under consideration; in particular, asymptotic fluctuation results, such as the Law of the Iterated Logarithm, carry over.

Proof of the ASIP: an outline

The salient trait of the proof is the *symbolic coding* of the geodesic flow, which allows to interpret it as a *suspension flow* over a Markov shift.



Single-orbit dynamics: borrowing from number theory

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The archetypical result in this direction is a celebrated theorem of Erdös and Kac, concerning the distribution of the arithmetic function $\omega \colon \mathbb{N}_{\geq 1} \to \mathbb{N}$,

$$\omega(n) := \text{ number of distinct prime divisors of } n.$$

The Erdös-Kac theorem and the temporal DLT

Theorem (Erdös-Kac, 1939)

The random variables $Y_N \colon \{1, \ldots, N\} \to \mathbb{R}$ *defined as*

$$Y_N(n) := \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}}, \; 1 \leq n \leq N, \quad n \text{ sampled uniformly},$$

converge in distribution towards $\mathcal{N}(0,1)$ *as* $N \to \infty$.

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Motivated by this statement, we introduce a temporal version of the CLT:

Definition (Temporal DLT)

The ergodic integrals $I_t(f,x)$ of an integrable function f satisfy a **temporal DLT** *along the orbit of* $x \in X$ if there is a non-trivial r.v. Y and families of real numbers $(A_T(x)), (B_T(x))$ with $B_T(x) \xrightarrow{T\to\infty} \infty$ such that $X_T(t) := \frac{I_t(f,x) - A_T(x)}{B_T(x)}$ converge in law towards Y as $t \sim \mathcal{U}_{[0,T]}$.

ASIP and temporal DLT

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for *any* r.v. Y there is an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of times such that

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This stems from the analogous property of *occupational* (random) *measures* of Brownian paths: almost surely, the set of accumulation points of the family $\frac{1}{T}\int_0^T \delta_{B(t)}\,\mathrm{d}t$ (for the weak topology on $\mathcal{P}(\mathbb{R})$) is $\mathcal{P}(\mathbb{R})$.

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Therefore:

$$\text{ASIP} \implies \text{almost surely, no temporal DLT.}$$

A much studied cousin: the horocycle flow

The geodesic flow in negative curvature belongs to the broad and nowadays deeply understood class of *Anosov flows*, characterized by the property that the flow, the expanded and the contracted direction "fill up" the whole space. The ASIP (hence the spatial CLT) holds for any such flow.

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Consider, for each $x \in T^1 S$, the set (called *stable manifold* through x)

$$W^s(x) := \{y \in T^1 S : d(g_t y, g_t x) \stackrel{t \to +\infty}{\longrightarrow} 0\},$$

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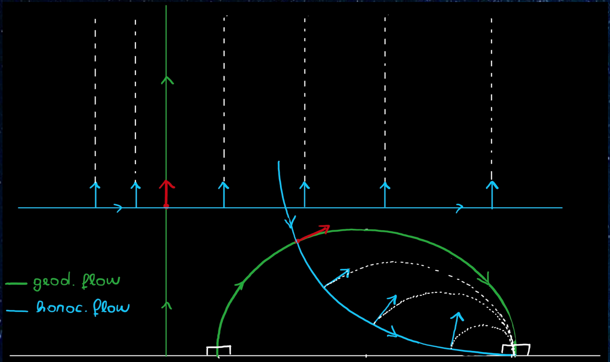
where d is a (any) Riemannian metric on T^1S. We might choose an arc-length parametrization $t \mapsto h_t(x)$ of the differentiable curve $W^s(x)$. The one-parameter *continuous* group $(h_t)_{t\in\mathbb{R}}$ is the horocycle flow on T^1S.

Geodesic and horocycle orbits on the hyperbolic plane

For the sake of illustration, we consider the standard model of *hyperbolic geometry* ($K \equiv -1$): the hyperbolic plane. As a smooth surface, it is the upper-half plane $\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\}$; the Riemannian metric is defined as $g_{x+iy}(v, w) = \frac{1}{y^2}\langle v, w\rangle$ for any $v, w \in T_{x+iy}\mathbb{H} \simeq \mathbb{R}^2$, so that

$$L(\gamma) = \int_a^b \frac{\|\gamma'(t)\|}{\mathrm{Im}\gamma(t)}\, \mathrm{d}t \quad \text{for any curve } \gamma\colon [a, b] \to \mathbb{H} \text{ of class } \mathrm{C}^1.$$



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Distributional limit theorems of various sorts are the subject of growing interest in dynamics; inherently motivated by the quest for universal limiting laws governing ergodic sums and integrals, they are part of a wholesale attempt to account for, and quantify, the manifestation of randomness in deterministic evolutions.