DISJOINTNESS OF HIGHER RANK DIAGONALIZABLE ACTIONS ON SEMISIMPLE AND SOLVABLE QUOTIENTS

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ABSTRACT. We prove that higher rank abelian actions by diagonalizable elements on S-arithmetic quotients of semisimple and solvable groups are disjoint in the sense of Furstenberg.

CONTENTS

1. INTRODUCTION

Furstenberg's influential paper [24] marked the outset of the study of rigidity properties of multiparameter algebraic actions on homogeneous spaces. His classification of closed subsets of the one-dimensional torus which are invariant under the action of a non-lacunary semigroup of positive integers represents, historically, the first discovery of the manifestation of a striking dichotomy between the individual and the global behaviour of a vast class of such actions: while the action of a single element of the acting group displays a certain flexibility, dynamically embodied by the existence of several invariant subsets and measures, the full action exhibits remarkable rigidity phenomena. Conjectures in this direction, in the context of higher-rank abelian actions, were formulated by Katok and Spatzier [33] and later by Margulis [41]; substantial progress has been made on the subject, leading mostly to a complete classification of invariant measures for the whole action under various positive entropy assumptions (see [8, 9, 15, 16, 33, 38]). Meanwhile, the study of the dynamical features of such actions on homogeneous spaces unveiled significant applications to number theory and arithmetic quantum chaos: the reader is referred to the survey [39] as well as to [10, 38] for more details thereupon.

In this paper, we are concerned with one particular aspect of the aforementioned rigidity, which might be loosely phrased as the absence of common dynamical properties between higher-rank diagonalizable actions on quotients of semisimple algebraic groups on one side, and the same type of actions on quotients of solvable groups on the other. The lack of a shared structure between the two systems is expressed in terms of the triviality of possible joinings thereof. In the same paper mentioned earlier ([24]), Furstenberg introduced the notion of disjointness for

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two measurable dynamical systems, in an attempt to give a precise meaning to the condition of them being relatively prime, in a sense suggested by the analogous condition of two integers in ordinary arithmetics. Disjointness, and the related notion of joinings, have played a pivotal role in the development of modern ergodic theory ever since; for a comprehensive account of the usefulness of joinings in the study of measurable dynamics, we refer the reader to [25]. More specifically, in the setting of homogenous dynamics, various results establishing scarcity of joinings have been established over the course of the last four decades [11,13,17,18,20,32,55].

We recall that, given two actions of a (abstract) group R on probability measure spaces $(X,\mathcal{A},\mu)$, $(Y,\mathcal{B},\nu)$ by measure-preserving transformations, a *joining* of the two actions is defined as a probability measure ρ on the product measurable space $(X\times Y,\mathcal{A}\otimes\mathcal{B})$ which is invariant for the induced diagonal R-action on $X\times Y$ and projects to μ and ν, respectively, under the canonical factor maps (the reader is referred to [21,69] for the required ergodic-theoretic background). The set of all joinings is always non-empty, as the product measure $\mu\times\nu$ obviously satisfies the criteria in the definition. The actions of R on X and Y are called *disjoint* if there are no joinings except for the trivial one, namely the product measure. Disjointness, at least for sufficiently regular continuous actions of a topological group R, implies that the two systems cannot admit a non-trivial common measurable factor, that is, a probability measure space $(Z,\mathcal{C},\lambda)$ with $0<\lambda(A)<1$ for some $A\in\mathcal{C}$, endowed with a measure-preserving action of R and measure-preserving factor maps $\phi_X\colon X\to Z$, $\phi_Y\colon Y\to Z$ which are equivariant with respect to the R-actions. Indeed, the existence of such a non-trivial common factor would give rise to the relatively independent joining $\mu\times_\lambda\nu$, which is concentrated on the set of points $(x,y)\in X\times Y$ for which $\phi_X(x)=\phi_Y(y)$ (see [21], Chap. 6), and as such is non-trivial.

1.1. **Setup and main result.** In this manuscript, we deal with joinings of abelian algebraic actions by diagonalizable elements on homogeneous spaces arising from perfect and solvable groups. In view of its recurring appearance in number-theoretical applications of homogeneous dynamics, we choose to work in the S-arithmetic context, namely over products of real and p-adic algebraic groups. We now set out to introduce the precise setup, referring to Sections 2, 3 and 5 for precise definitions of all the involved notions. Sections 3 and 4 include an extensive discussion of the role played by the various assumptions we shall consider in the statement of our main result.

In the sequel, assume $\mathbf{G}$ and $\mathbf{B}$ are, respectively, a perfect and a solvable Zariski-connected linear algebraic group defined over $\mathbb{Q}$, and let S be a fixed finite set of places of $\mathbb{Q}$ containing the infinite place. We denote by $\mathbf{G}(\mathbb{Q}_S)$ and $\mathbf{B}(\mathbb{Q}_S)$ the groups of $\mathbb{Q}_S$-points of $\mathbf{G}$ and $\mathbf{B}$, respectively, and let $G<\mathbf{G}(\mathbb{Q}_S)$, $B<\mathbf{B}(\mathbb{Q}_S)$ be closed subgroups of finite index[1].

If $\mathbf{H}$ is an algebraic group defined over $\mathbb{Q}$ and $H<\mathbf{H}(\mathbb{Q}_S)$ is a closed subgroup, we shall indicate with $H^{(1)}$ the group of S-units in H, that is

$$H^{(1)}=\left\{h=(h_\sigma)_{\sigma\in S}\in H:\forall\chi\in X_{\mathbb{Q}}(\mathbf{H})\ \prod_{\sigma\in S}|\chi(h_\sigma)|_\sigma=1\right\},$$

where $X_{\mathbb{Q}}(\mathbf{H})$ is the group of $\mathbb{Q}$-characters of $\mathbf{H}$, and the notation $|\cdot|_\sigma$ stands for the σ-adic absolute value on $\mathbb{Q}_\sigma$, for any place $\sigma\in S$.

Denote by $\mathcal{O}_S$ the subring of S-integral elements of $\mathbb{Q}$. The group $\mathbf{G}(\mathcal{O}_S)$ of S-integral points of $\mathbf{G}$ embeds diagonally in $\mathbf{G}(\mathbb{Q}_S)$ with discrete image. As the group of $\mathbb{Q}$-characters of the perfect group $\mathbf{G}$ is trivial, a celebrated result of Borel and Harish-Chandra ([4]) affirms that $\mathbf{G}(\mathcal{O}_S)$ actually embeds as a lattice in $\mathbf{G}(\mathbb{Q}_S)$. Similarly, we shall regard $\mathbf{B}(\mathcal{O}_S)$ as diagonally embedded in $\mathbf{B}(\mathbb{Q}_S)$ as a discrete subgroup; in this case the image need not be a lattice, in general. Let $\Gamma<G\cap\mathbf{G}(\mathbb{Q})$ and $\Lambda<B\cap\mathbf{B}(\mathbb{Q})$ be S-arithmetic groups, namely subgroups of $\mathbf{G}(\mathbb{Q})$ and $\mathbf{B}(\mathbb{Q})$ which are commensurable to $\mathbf{G}(\mathcal{O}_S)$ and $\mathbf{B}(\mathcal{O}_S)$, respectively; with $X=\Gamma\backslash G$

[1] Henceforth, a finite-index subgroup of a topological group R is always understood to be closed (hence open).

and $Y = \Lambda\backslash B$ we indicate the homogeneous spaces defined by Γ and Λ. The group G acts on X on the left by right translations via $g_0 \cdot \Gamma g = \Gamma g g_0^{-1}$ for any $g_0, g \in G$, and similarly for B.

As Γ is commensurable with $\mathbf{G}(\mathcal{O}_S)$, the quotient X comes equipped with a G-invariant Borel probability measure m_X, henceforth referred to as the Haar-Siegel measure, or the Haar measure for short, on X. We shall assume that X is *saturated by unipotents* (see [17, Def. 1.1]): the subgroup of G generated by all the unipotent elements in $\mathbf{G}(\mathbb{Q}_S)$ acts ergodically on X with respect to the Haar-Siegel measure[2].

Let $d \geq 2$ be an integer, and consider two group homomorphisms $a_G\colon \mathbb{Z}^d \to G$ and $a_B\colon \mathbb{Z}^d \to B$ with diagonalizable images, by which we mean that, for any element $a = (a_\sigma)_{\sigma\in S}$ in $a_G(\mathbb{Z}^d) \cup a_B(\mathbb{Z}^d)$, a_σ is diagonalizable over the algebraic closure $\overline{\mathbb{Q}_\sigma}$, for any $\sigma \in S$. Assume that the image of the morphism a_B is contained in the discrete subgroup Λ. Additionally, we impose two conditions on the homomorphism a_G, one of algebraic and another of topological flavour. Specifically, we assume on one hand that the image subgroup $a_G(\mathbb{Z}^d)$ consists of class-$\mathcal{A}'$ elements; our definition of a class-$\mathcal{A}'$ element (for which we refer to Section 3.2) was first formulated by Einsiedler and Lindenstrauss in [17], and extends the class-$\mathcal{A}$ notion considered earlier by Margulis and Tomanov in [42, 43]. On the other hand, we require that the projection of a_G to the $\mathbb{Q}_S$-points of every $\mathbb{Q}$-almost simple factor of $\mathbf{G}$ is topologically a proper map.

The group $\mathbb{Z}^d$ acts by homeomorphisms on X and Y by precomposing the canonical actions by right translations of G and B on their respective quotients with the homomorphisms a_G and a_B. The Haar-Siegel measure m_X is obviously invariant under the $\mathbb{Z}^d$-action. We equip the solvable quotient Y with a $\mathbb{Z}^d$-invariant measure m_Y of maximal entropy[3] with respect to the action of the subgroup $a_B(\mathbb{Z}^d)$: specifically, we assume that for every $\mathbf{n} \in \mathbb{Z}^d$ the entropy of m_Y for the transformation induced by $a_B(\mathbf{n})$ on Y equals the negative logarithm of the modulus of the adjoint automorphism induced by $a_B(\mathbf{n})$ on the Lie algebra of B, restricted to the $a_B(\mathbf{n})$-contracted eigenspaces. For a precise formulation of the condition, and an explanation of the terminology we adopt, the reader is referred to Section 3.3; relevant examples of such measures include the Haar-Siegel measure on the finite-volume homogeneous subspace $\Lambda\backslash\mathbf{B}(\mathbb{Q}_S)^{(1)} \subset Y$ and, more generally, any $a_B(\mathbb{Z}^d)$-invariant probability measure on Y which is also invariant under the $\mathbb{Q}_S$-points of the unipotent radical $R_{\mathrm{u}}(\mathbf{B})$ of $\mathbf{B}$. We further suppose that finite-index subgroups of $\mathbb{Z}^d$ act ergodically with respect to m_Y.

In the spirit of Ratner's measure classification theorem for unipotent actions [57], and in accordance with the analogous rigidity conjectures for diagonalizable actions formulated in [33] and [41], it is natural to expect joinings of such actions be of algebraic nature. In [17], the authors show that any ergodic joining of class-$\mathcal{A}'$ actions on perfect quotients saturated by unipotents is a $\mathbb{Q}$-algebraic measure (cf. [17, Def. 1.2]). The same conclusion holds in the context of actions by unipotent elements, as shown by Tomanov [67] in a refinement of the S-arithmetic extensions of Ratner's results [42, 58]. Translated into our context, this amounts to the existence of a $\mathbb{Q}$-subgroup $\mathbf{L} < \mathbf{G} \times \mathbf{B}$ and a finite-index subgroup $L < \mathbf{L}^{(1)}(\mathbb{Q}_S)$ such that μ is the unique uniform measure supported on a closed orbit of a conjugate of L.

Suppose, as it is the case in many relevant applications, that m_Y is a $\mathbb{Q}$-algebraic measure supported on a translated orbit $\Lambda B_1 b$ of a finite-index subgroup $B_1 < \mathbf{B}_1(\mathbb{Q}_S)$, where $\mathbf{B}_1$ is a $\mathbb{Q}$-subgroup of $\mathbf{B}$ containg $R_{\mathrm{u}}(\mathbf{B})$. If μ is a $\mathbb{Q}$-algebraic joining of m_X and m_Y, it necessarily follows that the $\mathbb{Q}$-subgroup $\mathbf{L}$ projects surjectively onto $\mathbf{G}$ and $\mathbf{B}_1$ (cf. Proposition 8.1); since $\mathbf{G}$ is

[2]This is only a mildly restrictive condition: if the solvable radical and the unipotent radical $R_{\mathrm{u}}(\mathbf{G})$ of $\mathbf{G}$ coincide (which is the case for a perfect group $\mathbf{G}$), and the quotient $\mathbf{G}/R_{\mathrm{u}}(\mathbf{G})$ does not admit any $\mathbb{Q}$-simple factor $\mathbf{G}_1$ for which $\mathbf{G}_1(\mathbb{Q}_S)$ is a compact group, there exists a finite-index normal subgroup $G_1 < G$ such that the S-arithmetic quotient $(\Gamma \cap G_1)\backslash G_1$ is saturated by unipotents (see [17, Rmk. 3.2]). Observe that the group generated by all the unipotent elements in $\mathbf{G}(\mathbb{Q}_S)$ is contained in G, as we shall clarify in Lemma 2.5.

[3]The proof of our main result shows that maximality of entropy is not strictly necessary for the theorems to hold as stated. We are currently planning to generalize the results accordingly in a subsequent paper.

perfect and $\mathbf{B}$ is solvable, they admit no non-trivial isomorphic quotients by normal subgroups, hence Goursat's lemma (see Proposition 8.2) forces $\mathbf{L} = \mathbf{G} \times \mathbf{B}_1$. Ignoring finite-index issues for the sake of illustration, it follows that the only $\mathbb{Q}$-algebraic joining possibly arising in this setup is the product measure $m_X \times m_Y$.

Accordingly, the chief goal of this article is to prove disjointness of the two measure-preserving $\mathbb{Z}^d$-actions on X and Y; in the case of a $\mathbb{Q}$-algebraic measure m_Y, this is equivalent, as just argued and up to finite-index, to asserting that every ergodic joining is $\mathbb{Q}$-algebraic. We establish the following theorem:

Theorem 1.1. *Let $\mathbf{G}$ and $\mathbf{B}$ be, respectively, a perfect and a solvable Zariski-connected linear algebraic group defined over $\mathbb{Q}$, S a finite set of places of $\mathbb{Q}$ containing the infinite place, $G < \mathbf{G}(\mathbb{Q}_S)$ and $B < \mathbf{B}(\mathbb{Q}_S)$ finite-index subgroups. Let $\Gamma < G \cap \mathbf{G}(\mathbb{Q})$, $\Lambda < B \cap \mathbf{B}(\mathbb{Q})$ be S-arithmetic subgroups, and denote by $X = \Gamma\backslash G$, $Y = \Lambda\backslash B$ the respective homogeneous spaces. Let $d \geq 2$ be an integer, and consider diagonalizable homomorphisms $a_G\colon \mathbb{Z}^d \to G$ and $a_B\colon \mathbb{Z}^d \to \Lambda$. Suppose a_G is subject to the following conditions:*

1. *the subgroup $a_G(\mathbb{Z}^d) < G$ is of class-$\mathcal{A}'$;*
2. *for any $\mathbb{Q}$-almost simple factor $\mathbf{G}_{\mathrm{s}}$ of $\mathbf{G}$, the projection of a_G to the group of $\mathbb{Q}_S$-points $\mathbf{G}_{\mathrm{s}}(\mathbb{Q}_S)$ is topologically a proper map.*

Let m_X be the Haar-Siegel measure on X and let m_Y be a $\mathbb{Z}^d$-invariant probability measure on Y with maximal entropy with respect to the action of the group $a_B(\mathbb{Z}^d)$. Assume X is saturated by unipotents, and finite-index subgroups of $\mathbb{Z}^d$ act ergodically on (Y, m_Y).

If μ is a $\mathbb{Z}^d$-invariant and ergodic joining of the measure-preserving actions of $\mathbb{Z}^d$ on (X, m_X) and (Y, m_Y), then μ is trivial, that is, μ equals the product measure $m_X \times m_Y$. As a consequence, the two $\mathbb{Z}^d$-actions are disjoint.

Remark 1.2. As a consequence of the combination of the two assumptions on the homomorphism a_G, finite-index subgroups of $\mathbb{Z}^d$ act ergodically on the measure space (X, m_X) as well. A straightforward adaptation of the proof of Lemma 4.8 justifies this claim.

In particular, the last statement of Theorem 1.1 follows follows from the first assertion by the following classical ergodic-decomposition argument: if μ is a joining of the measure-preserving actions of $\mathbb{Z}^d$ on (X, m_X) and (Y, m_Y), then choose a $\mathbb{Z}^d$-ergodic decomposition (cf. Section 4.1) $\mu = \int_Z \mu_z \,\mathrm{d}\rho(z)$, where $(Z, \mathcal{C}, \rho)$ is an auxiliary probability measure space. Then, since $\mathbb{Z}^d$ acts ergodically on (X, m_X) and (Y, m_Y), uniqueness of the ergodic decomposition implies that μ_z is a $\mathbb{Z}^d$-invariant ergodic joining of m_X and m_Y for ρ-almost every $z \in Z$ (cf. [21, Lem. 6.8]). Therefore, $\mu_z = m_X \times m_Y$ for ρ-almost every $z \in Z$ by the first assertion in Theorem 1.1, whence $\mu = m_X \times m_Y$.

The bulk of the work lies in the proof of a more restrictive version of Theorem 1.1, in which additional conditions regarding the eigenvalues of the acting elements are imposed. The statement reads as follows:

Theorem 1.3. *Let $\mathbf{G}$ and $\mathbf{B}$ be, respectively, a perfect and a solvable Zariski-connected linear algebraic group defined over $\mathbb{Q}$, S a finite set of places of $\mathbb{Q}$ containing the infinite place, $G < \mathbf{G}(\mathbb{Q}_S)$ and $B < \mathbf{B}(\mathbb{Q}_S)$ finite-index subgroups. Let $\Gamma < G \cap \mathbf{G}(\mathbb{Q})$, $\Lambda < B \cap \mathbf{B}(\mathbb{Q})$ be S-arithmetic subgroups, and denote by $X = \Gamma\backslash G$, $Y = \Lambda\backslash B$ the respective homogeneous spaces. Let $d \geq 2$ be an integer, and consider homomorphisms $a_G\colon \mathbb{Z}^d \to G$ and $a_B\colon \mathbb{Z}^d \to \Lambda$ such that the product homomorphism $a_G \times a_B\colon \mathbb{Z}^d \to G \times B$ is of class-$\mathcal{A}'$ and, for any $\mathbb{Q}$-almost simple factor $\mathbf{G}_{\mathrm{s}}$ of $\mathbf{G}$, the projection of a_G to the $\mathbb{Q}_S$-points $\mathbf{G}_{\mathrm{s}}(\mathbb{Q}_S)$ is a proper map.*

Suppose m_X and m_Y are probability measures on X and Y, respectively, satisfying the same assumptions as in Theorem 1.1.

If μ is a $\mathbb{Z}^d$-invariant and ergodic joining of the measure-preserving actions of $\mathbb{Z}^d$ on (X, m_X) and (Y, m_Y), then μ is trivial, that is, μ equals the product measure $m_X \times m_Y$. Hence, the two $\mathbb{Z}^d$-actions are disjoint.

We precede an overview of the proof of Theorems 1.1 and 1.3 by a discussion of the motivations underlying the present work.

1.2. **Disjointness and joint equidistribution of primitive rational points.** The results set forth in this article were motivated by the work of Einsiedler, Shah and the second named author in [19], concerning joint equidistribution of primitive rational points on the product of the two-dimensional torus with the unit tangent bundle of the modular surface. For the sake of illustration, we provide a short description of the relevant results established therein, which typify those obtained in the present work.

It was shown by Sarnak in [62] that closed horocycle orbits equidistribute, as the period of the orbit goes to infinity, in the homogeneous space $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$, which can be identified with the unit tangent bundle of the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ ([21, Chap. 9]). More precisely, denote

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ t\in\mathbb{R}, \quad v_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},\ s\in\mathbb{R}, \quad a_y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},\ y\in\mathbb{R}_{>0},$$

and the corresponding subgroups by $U = \{u_s : s\in\mathbb{R}\}, V = \{v_t : t\in\mathbb{R}\}, A = \{a_y : y \in \mathbb{R}_{>0}\}$. Then, for any continuous compactly supported function $f\colon\ \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$,

$$e^{-T}\int_0^{e^T} f(u_s\cdot \mathrm{SL}_2(\mathbb{Z})a_{e^{-T}})ds \overset{T\to+\infty}{\longrightarrow} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})} f\, d\mu\ ,$$

where μ is the probability Haar measure on $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$. For smooth functions, the statement can be upgraded to a quantitative estimate of the error. It is then natural to ask whether equidistribution carries over to sparser collections of points inside expanding horocycles. For instance, consider the sets of primitive rational points

$$\mathcal{R}_n = \big\{\mathrm{SL}_2(\mathbb{Z})u_{k/n}a_{y_n} : 0\le k\le n-1,\ \gcd(k,n)=1\big\},$$

where $(y_n)_{n\ge1}$ is a sequence of positive real numbers tending to zero. In general, the sets $\mathcal{P}_n$ do not distribute uniformly inside the space $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ as n goes to infinity, as obstructions may emerge: for instance, if $y_n = n^{-1}$ for every n, then an elementary computation shows that $\mathcal{P}_n$ is contained in the negative horocycle orbit $V\cdot \mathrm{SL}_2(\mathbb{Z})$, and actually the collection of pairs

$$\mathcal{P}_n^1 = \big\{\big(\mathrm{SL}_2(\mathbb{Z})u_{k/n}, \mathrm{SL}_2(\mathbb{Z})u_{k/n}a_{n^{-1}}\big) : 0\le k\le n-1, \gcd(k,n)=1\big\}$$

equidistributes in $(U\cdot \mathrm{SL}_2(\mathbb{Z}))\times(V\cdot \mathrm{SL}_2(\mathbb{Z}))$ towards the product of the uniform probability measures on the two orbits. Using Weyl's criterion, this follows immediately from well-known bounds on Kloosterman sums [37]. However, rescaling the sequence $(y_n)_n$ by appropriate negative powers allows to retain equidistribution; for instance, it is proved in [19] that the collection

$$\mathcal{P}_n^{1/2} = \big\{\big(k/n+\mathbb{Z}, \mathrm{SL}_2(\mathbb{Z})u_{k/n}a_{n^{-1/2}}\big) : 0\le k\le n-1, \gcd(k,n)=1\big\}$$

equidistributes in $(\mathbb{Z}\backslash\mathbb{R})\times(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}))$ towards the uniform probability measure $m_{\mathbb{T}}\times\mu$, where $\mathbb{T} = \mathbb{Z}\backslash\mathbb{R}$. This statement is then improved to show joint equidistribution of $\mathcal{P}_n^1$ and $\mathcal{P}_n^2$ under some congruence conditions (cf. [19, Thm. 1.3]): the collection

$$\mathcal{D}_n = \big\{\big(k/n+\mathbb{Z}, \overline{k}/n+\mathbb{Z}, \mathrm{SL}_2(\mathbb{Z})u_{k/n}a_{n^{-1/2}}\big) : 0\le k,\overline{k}\le n-1, k\overline{k}\equiv 1 \bmod n\big\}$$

equidistributes in $\mathbb{T}\times\mathbb{T}\times(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}))$ towards $m_{\mathbb{T}}\times m_{\mathbb{T}}\times\mu$, along sequences of integers n which are coprime to two fixed distinct prime numbers $p, q\in\mathbb{N}$. The key input for this joint equidistribution statement is a disjointness result for products of certain diagonalizable actions of the kind we consider in this manuscript, as we now briefly explain, referring to [19, Sec. 7] for the details. Let $S=\{p,q,\infty\}$ and denote $\mathbb{T}_S=\mathcal{O}_S\backslash\mathbb{Q}_S$ and $X_S=\mathrm{SL}_2(\mathcal{O}_S)\backslash\mathrm{SL}_2(\mathbb{Q}_S)$ the S-arithmetic extensions of $\mathbb{T}$ and $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$, respectively. The collection $\mathcal{D}_n$ lifts to the set of points

$$\mathcal{D}_n^{\mathrm{ext}}=\left\{\left(\mathcal{O}_S+k/n,\mathcal{O}_S+\overline{k}/n,\mathrm{SL}_2(\mathcal{O}_S)u_{k/n}a_{n^{-1/2}}\right):0\le k,\overline{k}\le n-1,k\overline{k}\equiv 1 \bmod n\right\}\subset\mathbb{T}_S^2\times X_S,$$

where $k/n,\overline{k}/n$ are considered as diagonally embedded in $\mathbb{Q}_S$, and likewise $u_{k/n},a_{n^{-1/2}}$ are diagonally embedded in $\mathrm{SL}_2(\mathbb{Q}_S)$; it then suffices to show that the sets $\mathcal{D}_n^{\mathrm{ext}},\gcd(n,pq)=1$, equidistribute with respect to the uniform measure on $\mathbb{T}_S^2\times X_S$. It turns out that any weak* limit ν of the uniform measures supported on such sets is invariant under the action of $\mathbb{Z}^2$ on $\mathbb{T}_S^2\times X_S$ given by

$$(a,b)\cdot(t,s,x)=(p^{2a}q^{2b}t,p^{-2a}q^{-2b}s,xa_{p^{-a}q^{-b}})\ ,\ (a,b)\in\mathbb{Z}^2,\ t,s\in\mathbb{T}_S,\ x\in X_S,$$

where $p^{2a}q^{2b},p^{-2a}q^{-2b}$ are diagonally embedded in $\mathbb{Q}_S$ and $a_{p^{-a}q^{-b}}$ stands for the diagonal embedding of the matrix $\begin{pmatrix}p^{-a}q^{-b}&0\\0&p^aq^b\end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Q}_S)$. What is more, ν projects onto the Haar measures $m_{\mathbb{T}_S^2},m_{X_S}$ on the factors, owing to individual equidistribution on each factor. It is thus a joining of the Haar measures for the projected $\mathbb{Z}^2$-actions on $\mathbb{T}_S^2$ and X_S. Invoking Theorem 1.1, of which [19, Prop. 7.5] and [19, Prop. 7.7] are special cases, for $\mathbf{G}=\mathrm{SL}_2$ and $\mathbf{B}=\mathbf{G}_m\ltimes\mathbf{G}_a$ (cf. Section 2.1 and 3.3) readily delivers $\nu=m_{\mathbb{T}_S^2}\times m_{X_S}$, as desired.

1.3. **Outline of the proof of Theorem 1.1.** We provide here a brief illustration of the argument leading to the proof of Theorem 1.1, referring the reader to Sections 4, 7 and 8 for a thorough treatment.

First, as we already alluded to, it is possible to reduce Theorem 1.1 to Theorem 1.3 by means of the decomposition of semisimple elements into their elliptic and non-compact parts. This is explained in Section 4.

In order to prove Theorem 1.3, a major ingredient is the following adaptation of Ratner's measure-rigidity results for unipotent actions (see [43], [17, Thm. 4.1] and [67, Thm. 2]), whose forumlation reveals the importance of reducing Theorem 1.1 to Theorem 1.3.

Proposition 1.4. *Let S be a finite set of places of $\mathbb{Q}$ containing the infinite place, $\mathbf{H}$ a Zariski-connected linear algebraic group defined over $\mathbb{Q}$, $H<\mathbf{H}(\mathbb{Q}_S)$ a finite-index subgroup, $\Delta<\mathbf{H}(\mathbb{Q})\cap H$ an S-arithmetic subgroup embedded diagonally in H, $X=\Delta\backslash H$ the quotient space. Assume $A<H$ is a commutative subgroup of class-$\mathcal{A}'$, $U<H$ an A-normalized Zariski-connected unipotent subgroup generated by one-parameter unipotent subgroups. Denote by M the closed subgroup of H generated by A and U.*

Let μ be an M-invariant and ergodic Borel probability measure on X, $\mu=\int_X\mu_x^{\mathcal{E}}\,\mathrm{d}\mu(x)$ a U-ergodic decomposition of μ given by a family of conditional measures for μ with respect to the σ-algebra $\mathcal{E}$ of U-invariant sets.

(1) There is a Zariski-connected $\mathbb{Q}$-subgroup $\mathbf{L}<\mathbf{H}$ of class $\mathcal{F}$ and an element $h\in H$ with $\Delta h\in\operatorname{supp}\mu$ such that μ is concentrated[4] on the orbit $\Delta N_H^1(\mathbf{L}(\mathbb{Q}_S))h$, where

$$N_H^1(\mathbf{L}(\mathbb{Q}_S))=\{h\in H: h \text{ normalizes } \mathbf{L}(\mathbb{Q}_S) \text{ and preserves the Haar measure } m_{\mathbf{L}(\mathbb{Q}_S)}\}.$$

(2) There is a finite-index subgroup $L<\mathbf{L}(\mathbb{Q}_S)$ such that the following hold:
(a) $h^{-1}Lh$ contains U and is normalized by M;
(b) for μ-almost every $x\in X$, the measure $\mu_x^{\mathcal{E}}$ is the unique $h^{-1}Lh$-invariant measure supported on the closed orbit $h^{-1}Lh\cdot x$.

[4]Here we mean that $\mu(\Delta N_H^1(\mathbf{L}(\mathbb{Q}_S))h)=1$; notice that orbits of closed subgroups are always Borel subsets, and the induced Borel structure turns them into standard Borel spaces (see [44, Thm. 2']).

Section 4.1 reviews the notions of conditional measures and ergodic decomposition. As far as subgroups of class-$\mathcal{F}$ are concerned, we adopt here the terminology introduced in [67]: a Zariski-connected $\mathbb{Q}$-subgroup $\mathbf{L} < \mathbf{H}$ is said to be of class $\mathcal{F}$ (relatively to the set of places S) if, for any proper normal $\mathbb{Q}$-subgroup $\mathbf{Q} < \mathbf{L}$, the group of $\mathbb{Q}_S$-points of the quotient $\mathbf{L}/\mathbf{Q}$ contains a unipotent element different from the identity.

It is convenient to introduce the following notation: H denotes the product group $G \times B$, which is a finite-index subgroup of $\mathbf{H}(\mathbb{Q}_S)$ for $\mathbf{H} = \mathbf{G} \times \mathbf{B}$, and Δ indicates the product $\Gamma \times \Lambda$, so that $\Gamma\backslash G \times \Lambda\backslash B$ and $\Delta\backslash H$ are isomorphic as topological H-spaces. Denoting by $a = a_G \times a_B \colon \mathbb{Z}^d \to H$ the diagonal homomorphism, we may thus interpret the joining μ as an A-invariant and ergodic probability measure on $\Delta\backslash H$, where $A = a(\mathbb{Z}^d)$.

For the time being, suppose we know that our $\mathbb{Z}^d$-invariant ergodic joining μ is U-invariant for some non-trivial Zariski-connected unipotent subgroup $U < H$ normalized by A and generated by one-parameter unipotent subgroups. In light of Proposition 1.4, we deduce that μ is, possibly upon a translation by an element of H, invariant under a finite-index subgroup of the group of $\mathbb{Q}_S$-points of a connected $\mathbb{Q}$-subgroup $\mathbf{L}$, and is concentrated on an orbit of the unit normalizer $N^1_H(\mathbf{L}(\mathbb{Q}_S))$. Since $\operatorname{supp}\mu$ contains an orbit of the group $N^1_H(\mathbf{L}(\mathbb{Q}_S))$, standard topological and algebraic arguments allow to deduce that the normalizer $N_{\mathbf{H}}(\mathbf{L})$ of $\mathbf{L}$ in $\mathbf{H}$ projects surjectively onto $\mathbf{G}$ on the perfect side. If $\mathbf{B}'$ denotes the projection of $N_{\mathbf{H}}(\mathbf{L})$ onto $\mathbf{B}$, then $\mathbf{G}$ and $\mathbf{B}'$ have no non-trivial isomorphic quotients in common, being a perfect and a solvable group, respectively. Goursat's lemma (cf. Proposition 8.2) forces $N_{\mathbf{H}}(\mathbf{L}) = \mathbf{G} \times \mathbf{B}'$, that is, $\mathbf{L}$ is a normal subgroup of $\mathbf{G} \times \mathbf{B}'$.

Now, if $\mathbf{G}$ is a $\mathbb{Q}$-simple group, $\mathbf{L}$ can be decomposed as a direct product $\mathbf{L}_1 \times \mathbf{L}_2$, where $\mathbf{L}_1 < \mathbf{G}$ and $\mathbf{L}_2 < R_{\mathrm{u}}(\mathbf{B})$ are Zariski-connected normal subgroups. If $\mathbf{L}_1$ is non-trivial, it follows readily that μ is the product measure (see Proposition 8.7); else, $\mathbf{L}_2$ is non-trivial and we may proceed by induction on the algebraic dimension of $R_{\mathrm{u}}(\mathbf{B})$, taking its quotient by $\mathbf{L}_2$. If $\mathbf{G}$ is semisimple, a similar decomposition $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$ holds; we resort to induction once more, this time on the number of $\mathbb{Q}$-simple factors of $\mathbf{G}$. Once the semisimple case is established, the case of a perfect group $\mathbf{G}$ follows by considering a Levi decomposition and applying the rigidity result in Proposition 8.10. All this is carried out in full detail in Section 8.

It remains to produce invariance under a non-trivial unipotent subgroup. For this we distinguish two separate cases, depending on whether the collections $\Psi(a_G)$ and $\Psi(a_B)$ of coarse Lyapunov weights (see Section 3.1 for their definition) are distinct or not. In the first case, assuming for instance that $[\alpha]$ is a coarse Lyapunov weight for a_G not contained in $\Psi(a_B)$, we apply [17, Prop. 6.5] and [16, Thm. 7.9] to obtain invariance of μ under the subgroup $U = G^{[\alpha]}$ (defined in Section 3.1), which by construction is non-trivial and satisfies the assumptions in Proposition 1.4. In the second case, additional invariance is obtained via either the high entropy method, as outlined for instance in [17, Sec. 7], or through a form of rigidity of the entropy function, adapted from more recent work by Einsiedler and Lindenstrauss on rigidity of higher rank actions on solenoids (cf. [18]).

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2. Preliminaries on algebraic and arithmetic groups

In this section, we introduce standard terminology and notation concerning linear algebraic groups, their Lie algebras and their S-arithmetic subgroups, which will be employed throughout the article. We mainly refer the reader to [2, 3, 40, 53, 65] for comprehensive treatments.

2.1. **Linear algebraic groups.** We first recall that a *place* of $\mathbb{Q}$ is an equivalence class of its completions (cf. [72, Chap. 3]). It is well-known that places of $\mathbb{Q}$ are in one-to-one correspondence with the set $P \cup \{\infty\}$, where $P = \{2, 3, \dots\}$ is the set of rational primes, and the correspondence is given by the embeddings of $\mathbb{Q}$ into $\mathbb{Q}_\infty := \mathbb{R}$ and into the fields of p-adic numbers $\mathbb{Q}_p$, for any $p \in P$. Throughout the manuscript, $S \subset P \cup \{\infty\}$ denotes a finite subset of places of $\mathbb{Q}$ containing the infinite place.

Let k be an algebraically closed field of characteristic zero. A linear algebraic group over k is a subgroup $\mathbf{G} < \mathrm{GL}_d(k)$ of invertible matrices which is defined by polynomial equations[5] with coefficients in k; more precisely, there exists a set of polynomials $T \subset k[X_{11}, \dots, X_{dd}]$ such that

$$\mathbf{G} = \{x = (x_{ij})_{1\le i,j\le d} \in \mathrm{GL}_d(k) : f(x_{11}, \dots, x_{dd}) = 0 \text{ for all } f \in T\}.$$

If $F \subset k$ is a subfield, we say that $\mathbf{G}$ is defined over F (or that G is an F-group) if T can be chosen inside $F[X_{11}, \dots, X_{dd}]$; in characteristic zero, this is equivalent to requiring that the complete ideal of relations of $\mathbf{G}$

$$\mathcal{I}(\mathbf{G}) = \{f \in k[X_{11}, \dots, X_{dd}] : f(x) = 0 \text{ for all } x \in \mathbf{G}\}$$

is generated by the F-submodule $\mathcal{I}_F(\mathbf{G}) := \mathcal{I}(\mathbf{G}) \cap F[X_{11}, \dots, X_{dd}]$. If $\mathbf{G}$ is defined over F, we denote by $\mathbf{G}(F) = \mathbf{G} \cap \mathrm{GL}_d(F)$ the subgroup of the F-points of $\mathbf{G}$.

We say that a linear algebraic group $\mathbf{G}$ is Zariski-connected (or simply connected) if it cannot be written as the union of two proper subsets, each of which is the zero locus of a family of polynomials with coefficients in k. Throughout the article, we shall tacitly assume that every linear algebraic group we deal with is Zariski-connected, unless otherwise specified.

Given two linear algebraic groups $\mathbf{G} < \mathrm{GL}_d(k)$ and $\mathbf{H} < \mathrm{GL}_{d'}(k)$ over the field k, it is straightforward to check that the product $\mathbf{G} \times \mathbf{H}$, canonically embedded via block-diagonal matrices in $\mathrm{GL}_{d+d'}(k)$, is a linear algebraic group over k. Furthermore, if $\mathbf{G}$ and $\mathbf{H}$ are Zariski-connected, then so is $\mathbf{G} \times \mathbf{H}$ [65, Thm. 1.5.4].

With $\mathbf{G}_{\mathrm{a}}$ and $\mathbf{G}_{\mathrm{m}}$ we shall denote, respectively, the k-algebraic groups given by the additive group of the field k and the multiplicative group $k^\times$ of its invertible elements.

If $\mathbf{G}$ is a linear algebraic group over k, an *algebraic subgroup* of $\mathbf{G}$ is a subgroup $\mathbf{H} < \mathbf{G}$ which is itself a linear algebraic group. We say that $\mathbf{H}$ is an F-subgroup if it is defined over F.

We say that a connected linear algebraic group $\mathbf{G} < \mathrm{GL}_d(k)$ is:

- perfect if $\mathbf{G}$ coincides with its own commutator subgroup $[\mathbf{G}, \mathbf{G}]$;
- solvable if it is solvable as an abstract group, meaning that the derived series $\mathbf{G}^0 = \mathbf{G}, \mathbf{G}^{i+1} = [\mathbf{G}^i, \mathbf{G}^i]$ for $i \ge 0$, terminates in the trivial subgroup;
- unipotent if it consists of unipotent elements, that is, for all $g \in \mathbf{G}$ there exists a positive integer n such that $(g - \mathbb{1}_d)^n = 0$, where $\mathbb{1}_d$ denotes the identity matrix in $\mathrm{GL}_d(k)$;
- diagonalizable if it is commutative and consists only of semisimple elements: for any $g \in \mathbf{G}$ there is $h \in \mathrm{GL}_d(k)$ such that hgh^{-1} is a diagonal matrix;
- simple if it does not contain any non-trivial, proper, Zariski-connected normal subgroup;

[5] We prefer working with this elementary definition of linear algebraic groups, rather than with the well-established, more general notion (cf. [3, 65]). This doesn't restrict the scope of our considerations, as all the arguments are unaffected by replacing the given linear algebraic groups with isomorphic copies; as is well-known (see, for instance, [65, Thm. 2.3.7]), every linear algebraic group, according to the abstract definition, is isomorphic to an algebraic subgroup of some GL_d.

- F-almost simple if it is defined over a subfield $F \leq k$ and does not contain any non-trivial, proper, Zariski-connected normal F-subgroup;
- reductive if it does not contain any non-trivial Zariski-connected unipotent normal subgroup;
- semisimple if it does not contain any non-trivial Zariski-connected solvable normal subgroup.

Given a linear algebraic group $\mathbf{G}$, the radical of $\mathbf{G}$, denoted by $R(\mathbf{G})$, is the maximal Zariski-connected solvable normal subgroup of $\mathbf{G}$; the unipotent radical of $\mathbf{G}$, indicated with $R_{\mathrm{u}}(\mathbf{G})$, is the maximal Zariski-connected unipotent normal subgroup of $\mathbf{G}$.

Let $\mathbf{G} < \mathrm{GL}_n$ be a linear algebraic group over k; for any element $g \in \mathbf{G}$, there is a unique pair $(g_{\mathrm{ss}}, g_{\mathrm{u}}) \in \mathbf{G}^2$ such that $g = g_{\mathrm{ss}} g_{\mathrm{u}} = g_{\mathrm{u}} g_{\mathrm{ss}}$, g_{ss} is semisimple and g_{u} is unipotent; this is called the multiplicative Jordan decomposition of g. The elements $g_{\mathrm{ss}}, g_{\mathrm{u}}$ are called, respectively, the semisimple and the unipotent part of g. If $\mathbf{G}' < \mathrm{GL}_m$ is another k-group and $\rho\colon \mathbf{G} \to \mathbf{G}'$ is a morphism of algebraic groups, then $\rho(g_{\mathrm{ss}}) = \rho(g)_{\mathrm{ss}}, \rho(g_{\mathrm{u}}) = \rho(g)_{\mathrm{u}}$ for any $g \in \mathbf{G}$.

Hereinafter, we shall confine ourselves to linear algebraic groups defined either over $\mathbb{Q}$ or over $\mathbb{Q}_\sigma$, for σ a place of $\mathbb{Q}$.

It is worth listing a couple of well-known structural results for linear algebraic groups, to which we shall repeatedly appeal. The first is the Levi decomposition:

Theorem 2.1 (cf. [53, Thm. 2.3])**.** *Let $\mathbf{G}$ be a Zariski-connected linear algebraic group defined over $\mathbb{Q}$, $R_{\mathrm{u}}(\mathbf{G})$ its unipotent radical. There exists a Zariski-connected, reductive $\mathbb{Q}$-subgroup $\mathbf{M}$ such that $\mathbf{G}$ is the semidirect product $\mathbf{M} \ltimes R_{\mathrm{u}}(\mathbf{G})$. Moreover, the commutator subgroup $[\mathbf{M}, \mathbf{M}]$ is a semisimple $\mathbb{Q}$-subgroup.*

Any subgroup $\mathbf{M}$ as in the stament of Theorem 2.1 is called a Levi factor of $\mathbf{G}$.

Next, we recall the structure theorem for semisimple groups:

Theorem 2.2 (cf. [53, Prop. 2.4])**.** *Let $\mathbf{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$, $(\mathbf{G}_i)_{i\in I}$ the collection of minimal non-trivial Zariski-connected normal $\mathbb{Q}$-subgroups of $\mathbf{G}$. Then I is a finite set and $\mathbf{G}$ is an almost direct product of the $\mathbf{G}_i$. In particular, $\mathbf{G}$ is an almost direct product of $\mathbb{Q}$-almost simple groups.*

Spelling out the statement, the product map $\prod_{i\in I} \mathbf{G}_i \to \mathbf{G}$ is a surjective morphism of algebraic groups (cf. [65, Chap. 2] for the definition) with finite kernel. The $\mathbf{G}_i$'s are called the $\mathbb{Q}$-almost simple factors of $\mathbf{G}$.

More generally, suppose that $\mathbf{G}$ is a perfect Zariski-connected $\mathbb{Q}$-group; if $\mathbf{G} = \mathbf{M} \ltimes R_{\mathrm{u}}(\mathbf{G})$ is a Levi decomposition as in Theorem 2.1, then $\mathbf{M} = [\mathbf{M}, \mathbf{M}]$ is semisimple. By a slight abuse of terminology, we shall refer to the $\mathbb{Q}$-almost simple factors of a given Levi factor $\mathbf{G}_{ss}$ of $\mathbf{G}$ as the $\mathbb{Q}$-almost simple factors of $\mathbf{G}$. Whenever we adopt this terminology, we thus assume implicitly that a choice of a Levi factor has been made in advance.

Let now $\mathbf{L}$ be a Zariski-connected normal $\mathbb{Q}$-subgroup of a Zariski-connected, semisimple $\mathbb{Q}$-group $\mathbf{G}$; if $\mathbf{G}_1, \dots, \mathbf{G}_r$ are the $\mathbb{Q}$-almost simple factors of $\mathbf{G}$, then the projection to $\mathbf{G}_i$ of the inverse image of $\mathbf{L}$ under the isogeny $\mathbf{G}_1 \times \cdots \times \mathbf{G}_r \to \mathbf{G}$ is a normal connected $\mathbb{Q}$-subgroup of the $\mathbb{Q}$-almost simple group $\mathbf{G}_i$, whence it is either the trivial group or the whole $\mathbf{G}_i$, for all $i = 1, \dots, r$. More is true, namely:

Corollary 2.3. *Let $\mathbf{G}$ be a Zariski-connected, semisimple $\mathbb{Q}$-group $\mathbf{G}$ with $\mathbb{Q}$-almost simple factors $\mathbf{G}_1, \dots, \mathbf{G}_r$, $\mathbf{L}$ a non-trivial, Zariski-connected normal $\mathbb{Q}$-subgroup. There is a unique subset $J \subset \{1, \dots, r\}$ such that $\mathbf{L}$ is the image of $\prod_{j\in J} \mathbf{G}_j$ under the canonical map $\prod_{i=1}^r \mathbf{G}_i \to \mathbf{G}$, and $\mathbf{L} \cap \mathbf{G}_i$ is finite for all $i \notin J$.*

2.2. **Lie algebras of linear algebraic groups.** We now intend to review the necessary background on Lie algebras of linear algebraic groups. Specifically, we aim to define and survey the basic properties of Lie algebras of groups of the form $G = \mathbf{G}(\mathbb{Q}_p)$, where $\mathbf{G}$ is a linear algebraic group defined over $\mathbb{Q}$ and p is a finite place of $\mathbb{Q}$. As for real Lie groups, these Lie algebras encode locally the structure of the group operations, and thus are peculiarly meaningful to the understanding of the sort of dynamics we are interested in. We also refer the reader to [29] for a more extensive treatment of the topic.

Let $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$ denote the set of square matrices of size d with coefficients in $\mathbb{Q}_p$, equipped with the unique Hausdorff topology making it into a topological vector space over $\mathbb{Q}_p$. We endow $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$ with the norm $\|x\|_p \coloneqq \sup_{1\leq i,j\leq d} |x_{i,j}|_p$ for any $x = (x_{i,j})_{1\leq i,j,\leq d} \in \mathrm{Mat}_{dd}(\mathbb{Q}_p)$, where $|\cdot|_p$ denotes, as before, the p-adic absolute value on $\mathbb{Q}_p$. Given $\varepsilon > 0$, we shall adopt the shorthand notation $(-\varepsilon, \varepsilon)_p = \{t \in \mathbb{Q}_p : |t|_p < \varepsilon\}$.

Definition 2.4. An *analytic curve* into $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$ is a map $\phi\colon (-\varepsilon, \varepsilon)_p \to \mathrm{Mat}_{dd}(\mathbb{Q}_p)$ such that

$$\phi(t) = \sum_{k=0}^{\infty} t^k x_k \ , \ t \in (-\varepsilon, \varepsilon)_p \ ,$$

where $x_k \in \mathrm{Mat}_{dd}(\mathbb{Q}_p)$ for all $k \geq 0$ and the series is absolutely convergent for all $t \in (-\varepsilon, \varepsilon)_p$.

Given an analytic curve ϕ into $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$, we define the *tangent* of ϕ to be $x_1 \in \mathrm{Mat}_{dd}(\mathbb{Q}_p)$, and write $\phi'(0) = x_1$. If $G < \mathrm{GL}_d(\mathbb{Q}_p)$ is a subgroup, by an analytic curve in G we mean an analytic curve ϕ such that $\phi(t) \in G$ for all $t \in (-\varepsilon, \varepsilon)_p$ and $\phi(0) = \mathbb{1}_d$. An analytic curve ϕ is called an analytic one-parameter subgroup if

$$\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) \text{ for all } t_1, t_2 \in (-\varepsilon, \varepsilon)_p \ .$$

Motivated by the analogous notion in the setting of real Lie groups, we define the Lie algebra $\mathfrak{g}$ of the group $G = \mathbf{G}(\mathbb{Q}_p)$ as the set of all tangents to analytic one-parameter subgroups in G. We claim that $\mathfrak{g}$ is indeed a Lie subalgebra of $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$, and for the sake of completeness we sketch the standard argument for this hereunder.

We recall that we may define an exponential map on the open ball of radius $r_p = p^{-\frac{1}{p-1}}$ around 0 in $\mathbb{Q}_p$ via the usual formula

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!} \ , \ |x|_p < p^{-\frac{1}{p-1}} \ . \tag{2.1}$$

Its inverse is given by the logarithm, defined on the open ball of radius r_p around 1 as

$$\log(1+x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n} \ , \ |x|_p < r_p \ .$$

Just as in the real case, these two functions may be extended to appropriate subsets of $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$. Specifically, if $r_p = p^{-\frac{1}{p-1}}$ denotes the radius of convergence of the exponential power series in (2.1), then, for any $x \in \mathrm{Mat}_{dd}(\mathbb{Q}_p)$ such that $\|x\|_p < r_p$, we set

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!} \ , \quad \log(1+x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n} \ ,$$

so that $\exp(\log(1+x)) = 1 + x$ and $\log(\exp(x)) = x$ for any $x \in \mathrm{Mat}_{dd}(\mathbb{Q}_p)$ with $\|x\|_p < r_p$, as follows from standard manipulation of formal power series.

By means of the exponential function, it is straightforward to show that we may equally consider tangents to arbitrary analytic curves in G to define its Lie algebra, instead of restricting ourselves to one-parameter subgroups; more precisely, we have that

$$\mathfrak{g} = \{x \in \mathrm{Mat}_{dd}(\mathbb{Q}_p) : \exists\, \phi\colon (-\varepsilon, \varepsilon)_p \to G \text{ analytic with } \phi'(0) = x\} \ . \tag{2.2}$$

Indeed, if $\phi\colon (-\varepsilon,\varepsilon)_p \to G$ is an analytic curve with tangent $\phi'(0) = x \in \mathrm{Mat}_{dd}(\mathbb{Q}_p)$, then an easy calculation shows that

$$x = \lim_{k\to\infty} p^{-k}\log\phi(p^k)\ . \tag{2.3}$$

Let $B_{r_p}(\mathbb{1}_d)$ denote the open ball of radius r_p around the identity matrix in $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$. Since $U = \log(B_{r_p}(\mathbb{1}_d)\cap G)$ is a compact subset of $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$, its cone $\mathcal{C}(U) = \{\lambda x : \lambda \in \mathbb{Q}_p, x \in U\}$ is closed. Thus, (2.3) implies that $x \in \mathcal{C}(U)$, so that $\lambda x \in U$ for some $\lambda \in \mathbb{Q}_p$. Therefore, the curve $\psi(t) = \exp(tx)$, defined for $|t|_p$ sufficiently small, is an analytic one-parameter subgroup in G with tangent x (we refer the reader to [29] for the details of this argument). The fact that $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$ is now a direct consequence of the equality in (2.2).

We hasten to add that the previous construction (and the ensuing observations) works equally well for any closed subgroup $G < \mathrm{GL}_d(\mathbb{Q}_p)$. Conversely, it turns out that for any Lie subalgebra $\mathfrak{g}$ of $\mathrm{Mat}_{dd}(\mathbb{Q}_p)$ there exists a closed subgroup $G < \mathrm{GL}_d(\mathbb{Q}_p)$ having $\mathfrak{g}$ as its Lie algebra in the above sense (actually, it suffices to define $G = \exp(\mathfrak{g}\cap B_{r_p}(0))$).

2.3. **Linear algebraic groups over local fields.** If $\mathbf{G} < \mathrm{GL}_d(\mathbb{C})$ is a linear algebraic group defined over $\mathbb{R}$, the group $\mathbf{G}(\mathbb{R})$ of its real points is equipped with the topology induced by the euclidean topology on $\mathbb{R}^{d^2}$, making it into a locally compact topological group.[6] Similarly, if $\mathbf{G}$ is a p-adic linear algebraic group (that is, $\mathbf{G} < \mathrm{GL}_d(\overline{\mathbb{Q}_p})$ for some algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$) defined over $\mathbb{Q}_p$, we equip $\mathbf{G}(\mathbb{Q}_p)$ with the subspace topology derived from the unique Hausdorff topology making $\mathbb{Q}_p^{d^2}$ into a topological $\mathbb{Q}_p$-vector space.

If σ is a place of $\mathbb{Q}$ and $\mathbf{G}$ is a $\mathbb{Q}_\sigma$-group, a one-parameter $\mathbb{Q}_\sigma$-subgroup of $\mathbf{G}$ is a morphism $\phi\colon \mathbf{G}_{\mathrm{a}} \to \mathbf{G}$ of algebraic groups defined over $\mathbb{Q}_\sigma$. It is called a one-parameter unipotent $\mathbb{Q}_\sigma$-subgroup if $\phi(\mathbf{G}_{\mathrm{a}})$ consists only of unipotent elements. Abusing terminology, we shall also refer to the image $\phi(\mathbf{G}_{\mathrm{a}})$ as a one-paramter unipotent subgroup; the intended meaning is each time clear from the context.

We record here the following statement about one-parameter unipotent subgroups.

Lemma 2.5. *Let σ be a place of $\mathbb{Q}$, $\mathbf{G} < \mathrm{GL}_d$ a $\mathbb{Q}_\sigma$-group, $g \in \mathbf{G}(\mathbb{Q}_\sigma)$ a unipotent element. Then g is contained in a one-parameter unipotent $\mathbb{Q}_\sigma$-subgroup $\mathbf{U}$ of $\mathbf{G}$. If F is a subfield of $\mathbb{Q}_\sigma$ and $g \in \mathbf{G}(F)$, then the one-parameter subgroup is defined over F. Moreover, if $G < \mathbf{G}(\mathbb{Q}_\sigma)$ is a finite-index subgroup, then $\mathbf{U}(\mathbb{Q}_\sigma) < G$.*

Proof. Let $X = \log g$. This is well-defined as g is unipotent. Note that $X \in \mathrm{Mat}_{dd}(F)$ if $g \in \mathbf{G}(F)$. Hence the group $\mathbf{U} < \mathrm{GL}_d$ given by the image of the morphism $\mathbf{G}_{\mathrm{a}} \to \mathrm{GL}_d$, $t \mapsto \exp(tX)$, is a closed subgroup defined over F. The group $\mathbf{U}$ is clearly one-dimensional. As $H = \{g^m : m \in \mathbb{Z}\}$ is infinite, the Zariski closure of H is at least one-dimensional[7]. $\mathbf{U}$ is a one-dimensional, Zariski-connected, Zariski-closed subgroup containing H, and hence $\mathbf{U}$ is the Zariski closure of H. As $\mathbf{G}$ is Zariski closed and $H < \mathbf{G}$, we get $\mathbf{U} < \mathbf{G}$. Finally, we get

$$\mathbf{U}(\mathbb{Q}_\sigma) = \mathbf{U}\cap \mathrm{GL}_d(\mathbb{Q}_\sigma) < \mathbf{G}\cap\mathrm{GL}_d(\mathbb{Q}_\sigma) = \mathbf{G}(\mathbb{Q}_\sigma).$$

Note that the Jordan normal form of X is defined over $\mathbb{Q}_\sigma$. Using this, it is easy to see that $\exp(tX) \in \mathrm{GL}_d(\mathbb{Q}_\sigma)$ if and only if $t \in \mathbb{Q}_\sigma$. In particular, $\mathbf{U}(\mathbb{Q}_\sigma)$ agrees with the image of the unipotent one-parameter subgroup $t \mapsto \exp(tX)$, $t \in \mathbb{Q}_\sigma$.

Assume now that $G < \mathbf{G}(\mathbb{Q}_\sigma)$ is a closed subgroup of finite index and suppose without loss of generality that G is normal in $\mathbf{G}(\mathbb{Q}_\sigma)$. By the second isomorphism theorem we have

$$\mathbf{U}(\mathbb{Q}_\sigma)/\big(\mathbf{U}(\mathbb{Q}_\sigma)\cap G\big) \cong \big(\mathbf{U}(\mathbb{Q}_\sigma)G\big)/G < \mathbf{G}(\mathbb{Q}_\sigma)/G.$$

[6]Unless otherwise specified, all topological spaces under considerations are assumed to be locally compact, Hausdorff and second countable.

[7]We refer to [65, Sec. 1.8] for the notion of dimension of an algebraic variety.

Hence $\mathbf{U}(\mathbb{Q}_\sigma) \cap G$ has finite index in $\mathbf{U}(\mathbb{Q}_\sigma)$. It therefore suffices to show that $\mathbb{Q}_\sigma$ does not have any non-trivial closed subgroup of finite index. If σ is infinite, this is immediate. Hence we assume that $\sigma = p$ is a finite prime. Let $L < \mathbb{Q}_p$ be a closed subgroup of finite index. As $\mathbb{Q}_p$ is abelian, L is a normal subgroup, and hence we obtain an isomorphism between the group of characters of $\mathbb{Q}_p/L$ and the unitary characters $\chi\colon \mathbb{Q}_p \to \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ satisfying $L < \ker\chi$. Any such character has finite range, which will give the desired contradiction. To this end, we recall that the Pontryagin dual (cf. [22]) $\widehat{\mathbb{Q}_p}$ is isomorphic to $\mathbb{Q}_p$ via the following explicit isomorphism. Fix a character $\chi_1\colon \mathbb{Q}_p \to \mathbb{S}^1$ which maps $t = \sum_{k=\ell}^{\infty} a_k p^k \in \mathbb{Q}_p$ to

$$\chi_1(t) = \exp\left(2\pi \mathrm{i} \sum_{k=\ell}^{-1} a_k p^k\right).$$

The isomorphism $\psi : \mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$ is then given by

$$\psi(a)(t) := \chi_1(at)\ , \quad a, t \in \mathbb{Q}_p.$$

From this it is easy to see that no non-trivial character of $\mathbb{Q}_p$ has finite range. Indeed, let $a \in \mathbb{Q}_p\backslash\{0\}$, then $\psi(a)(p^{-n}a^{-1})$ is a primitive p^n-th root of unity whenever $n \in \mathbb{N}$. In particular, the range of $\psi(a)$ contains all roots of unity, and hence is infinite. As $\widehat{\mathbb{Q}_p/L}$ separates points ([22, Thm. 3.34]), this implies that $\mathbb{Q}_p/L$ is trivial, that is, $L = \mathbb{Q}_p$. □

2.4. ***S*-algebraic groups and *S*-arithmetic subgroups.** The main reference for this subsection is [40]. In the sequel, a linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, or, concisely, an algebraic $\mathbb{Q}$-group, is always intended to be a subgroup of some $\mathrm{GL}_d(\overline{\mathbb{Q}})$; conforming to a well-established convention, we shall drop the reference to $\overline{\mathbb{Q}}$ and simply write GL_d when we refer to the full group of $\overline{\mathbb{Q}}$-points of GL_d.

If $\mathbf{G} < \mathrm{GL}_d$ is an algebraic $\mathbb{Q}$-group and σ is a place of $\mathbb{Q}$, there is an obvious way to identify $\mathbf{G}$ with a linear algebraic group over the field $\overline{\mathbb{Q}_\sigma}$; thus, we can consider the group of its $\mathbb{Q}_\sigma$-points defined as $\mathbf{G}(\mathbb{Q}_\sigma) := \mathbf{G}(\overline{\mathbb{Q}_\sigma}) \cap \mathrm{GL}_d(\mathbb{Q}_\sigma)$.

Given a finite set of places S of $\mathbb{Q}$ containing the infinite place, we denote by $\mathbb{Q}_S = \prod_{\sigma\in S} \mathbb{Q}_\sigma$; moreover, for each element $t = (t_\sigma)_{\sigma\in S} \in \mathbb{Q}_S$, we let $|t|_S := \prod_{\sigma\in S} |t_\sigma|_\sigma$.

If $\mathbf{G}$ an algebraic $\mathbb{Q}$-group, the notation $\mathbf{G}(\mathbb{Q}_S)$ stands for the set of its $\mathbb{Q}_S$-points, that is, $\mathbf{G}(\mathbb{Q}_S) := \prod_{\sigma\in S} \mathbf{G}(\mathbb{Q}_\sigma)$, which is a locally compact group for the product topology.

Definition 2.6. An *S-algebraic group* G is a finite-index subgroup of $\mathbf{G}(\mathbb{Q}_S)$, where $\mathbf{G}$ is an algebraic $\mathbb{Q}$-group and S is a finite set of places of $\mathbb{Q}$ containing the infinite place.

Given an S-algebraic group $G < \mathbf{G}(\mathbb{Q}_S)$, we define the *unit group* of G as the subgroup

$$G^{(1)} = \left\{ g = (g_\sigma)_{\sigma\in S} \in G : \forall\chi \in X_{\mathbb{Q}}(\mathbf{G}), \prod_{\sigma\in S} |\chi(g_\sigma)|_\sigma = 1 \right\},$$

where $X_{\mathbb{Q}}(\mathbf{G})$ denotes the group of $\mathbb{Q}$-characters of $\mathbf{G}$, that is, of morphisms of $\mathbb{Q}$-groups $\mathbf{G} \to \mathbf{G}_{\mathrm{m}}$ defined over $\mathbb{Q}$.

We now turn to the discussion of S-arithmetic subgroups and lattices. For the sake of completeness, we recall that a *lattice* in a locally compact group G is a discrete subgroup $\Gamma < G$ such that the topological space of right cosets $X = \Gamma\backslash G$ admits a Borel probability measure m_X, called the Haar-Siegel measure on X, which is invariant under the action of G on $\Gamma\backslash G$ by right translations $g \cdot \Gamma g_0 = \Gamma g_0 g^{-1}$, $g, g_0 \in G$; this means that $m_X(g \cdot A) = m_X(A)$ for any $g \in G$ and any Borel set $A \subset X$. If we equip G with a metric d_G inducing its topology and invariant under left translations,[8] there is a derived metric d_X inducing the quotient topology

[8] It is a theorem of G. Birkhoff and Kakutani that any Hausdorff, first countable group G admits such a metric, see [45, Sect. 1.22]

on X and defined by

$$d_X(\Gamma g, \Gamma h) = \inf_{\gamma_1,\gamma_2\in\Gamma} d_G(\gamma_1 g, \gamma_2 h) = \inf_{\gamma\in\Gamma} d_G(\gamma g, h) ,$$

for any $\Gamma g, \Gamma h \in X$, where the second equality follows from left-invariance of d_G.

For any set of places S of $\mathbb{Q}$, we define the ring of S-integers of $\mathbb{Q}$ as

$$\mathcal{O}_S = \{x \in \mathbb{Q} : |x|_p \le 1 \text{ for every finite place } p \notin S\} ;$$

in particular, if S is finite and $S \setminus \{\infty\} = \{p_1, \dots, p_r\}$, then $\mathcal{O}_S = \mathbb{Z}[p_1^{-1}, \dots, p_r^{-1}]$.

For any algebraic $\mathbb{Q}$-group $\mathbf{G} < \mathrm{GL}_d$, we may consider the subgroup of its $\mathcal{O}_S$-points, defined as $\mathbf{G}(\mathcal{O}_S) := \mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}_d(\mathcal{O}_S)$. The subgroup $\mathbf{G}(\mathcal{O}_S)$ embeds diagonally in the product group $\mathbf{G}(\mathbb{Q}_S)$; the image, which we shall identify with $\mathbf{G}(\mathcal{O}_S)$, is a discrete subgroup. It turns out that $\mathbf{G}(\mathcal{O}_S)$ is actually contained in the unit group $\mathbf{G}^{(1)}(\mathbb{Q}_S)$ of $\mathbf{G}(\mathbb{Q}_S)$, and it is a lattice in it: this follows combining [53, Thm. 5.6] and Proposition A.3 in the appendix.

Recall that, given an abstract group G, two subgroups H_1, H_2 of G are called commensurable if their intersection $H_1 \cap H_2$ is of finite index both in H_1 and H_2. In the following definition, both $\mathbf{G}(\mathcal{O}_S)$ and $\mathbf{G}(\mathbb{Q})$ are identified with their diagonal embeddings in $\mathbf{G}(\mathbb{Q}_S)$.

Definition 2.7. Given an S-algebraic group $G < \mathbf{G}(\mathbb{Q}_S)$, an *$S$-arithmetic subgroup* is a subgroup $\Gamma < G \cap \mathbf{G}(\mathbb{Q})$ which is commensurable to the subgroup of S-integral points $\mathbf{G}(\mathcal{O}_S)$

Notice that, for every S-arithmetic subgroup Γ of an S-algebraic group $G < \mathbf{G}(\mathbb{Q}_S)$, the intersection $\Gamma \cap G^{(1)}$ is a lattice in the unit group $G^{(1)}$; in particular, if $\mathbf{G}$ has no non-trivial $\mathbb{Q}$-characters (which is the case, for instance, if $\mathbf{G}$ is perfect) then Γ is a lattice in G ([4]).

Henceforth, we shall employ the term *S-arithmetic quotient* to refer to a homogeneous space of the form $X = \Gamma\backslash G$, where G is an S-algebraic group and $\Gamma < G$ is an S-arithmetic subgroup.

3. Generalities on diagonalizable actions

3.1. **Lyapunov weights and (un-)stable leaves.** For this subsection, we mostly follow [9, Sec. 4]. Let $G < \mathbf{G}(\mathbb{Q}_S)$ be an S-algebraic group; we define the Lie algebra of G as $\mathfrak{g} := \bigoplus_{\sigma\in S} \mathfrak{g}_\sigma$, where $\mathfrak{g}_\sigma$ is the Lie algebra of $\mathbf{G}(\mathbb{Q}_\sigma)$ as defined in Section 2.2. It is endowed with a canonical structure of $\mathbb{Q}_S$-module. By abuse of language, we shall refer to a $\mathbb{Q}_S$-submodule of $\mathfrak{g}$ as a subspace.

For convenience of the reader, we include the following result which will be needed in the forthcoming discussion on characters.

Lemma 3.1. *If $V \subset \mathfrak{g}$ is a subspace, then it decomposes as a direct sum $V = \bigoplus_{\sigma\in S} V_\sigma$, where $V_\sigma \subset \mathfrak{g}_\sigma$ is the subspace defined by the projection of V to $\mathfrak{g}_\sigma$.*

Proof. Given $\sigma \in S$, let $\delta_\sigma \in \mathbb{Q}_S$ denote the element with components $(\delta_\sigma)_\tau$ equal to 1 if $\sigma = \tau$ and 0 otherwise. Every element $v \in V$ satisfies $v = \sum_{\sigma\in S} \delta_\sigma v$ with $\delta_\sigma v \in \mathfrak{g}_\sigma$. This implies in particular that $\bigoplus_{\sigma\in S} V_\sigma \subseteq V$. The opposite inclusion is clear. □

Denote by Ad the adjoint action of G on its Lie algebra $\mathfrak{g}$, which is defined componentwise as $\mathrm{Ad}_g(x) = gxg^{-1}$ for any $g \in G, x \in \mathfrak{g}$. Consider now a diagonalizable homomorphism $a\colon \mathbb{Z}^d \to G$; then $\mathfrak{g}$ decomposes as the direct sum of subspaces which are invariant under the action of any automorphism $\mathrm{Ad}_{a(\mathbf{n})}, \mathbf{n} \in \mathbb{Z}^d$. More precisely, let $\overline{\mathbb{Q}_S}$ denote the product of the algebraic closures $\overline{\mathbb{Q}_\sigma}$, $\sigma \in S$, and $\overline{\mathbb{Q}_S}^\times$ be the set of its invertible elements; there exist finitely many characters $\chi\colon \mathbb{Z}^d \to \overline{\mathbb{Q}_S}^\times$ and non-trivial subspaces $\mathfrak{g}^\chi$ such that $\mathfrak{g} = \bigoplus_\chi \mathfrak{g}^\chi$ and

$$(\mathrm{Ad} \circ a)(\mathbf{n})x = \chi(\mathbf{n})x , \quad x \in \mathfrak{g}^\chi \otimes \overline{\mathbb{Q}_S}, \ \mathbf{n} \in \mathbb{Z}^d .$$

For each $\sigma \in S$, let χ_σ be the projection of χ to $\overline{\mathbb{Q}_\sigma}$. As a consequence of Lemma 3.1, we may always decompose a character arising, as before, as an eigenvalue of the homomorphism a into a product $\chi = \prod_{\sigma\in S}\chi_\sigma$, where $\chi_\sigma\colon \mathbb{Z}^d \to \overline{\mathbb{Q}_\sigma}^\times$ is either trivial or an eigenvalue of the projection of a onto $\mathbf{G}(\overline{\mathbb{Q}_\sigma})$ acting on $\mathfrak{g}_\sigma \otimes \overline{\mathbb{Q}_\sigma}$. Henceforth, we shall call *characters of a* those characters χ_σ arising from eigenvalue-characters $\chi\colon \mathbb{Z}^d \to \overline{\mathbb{Q}_S}^\times$. It will be convenient to consider χ_σ as a $\overline{\mathbb{Q}_S}$-valued function by setting $(\chi_\sigma(\mathbf{n}))_{\sigma'} = 1$ for all $\mathbf{n} \in \mathbb{Z}^d$ and for all $\sigma' \in S \setminus \{\sigma\}$.

Recall that the absolute value $|\cdot|_\sigma$ admits a unique extension to an absolute value on the algebraic closure $\overline{\mathbb{Q}_\sigma}$ ([60, Chap. 3]). We shall adopt the same notation $|\cdot|_\sigma$ for such extension.

Definition 3.2. A *Lyapunov weight* of a is a functional $\alpha\colon \mathbb{Z}^d \to \mathbb{R}$ such that there exists $\sigma \in S$ and a character $\chi_\sigma\colon \mathbb{Z}^d \to \overline{\mathbb{Q}_\sigma}^\times$ of a satisfying $\alpha(\mathbf{n}) = \log|\chi_\sigma(\mathbf{n})|_\sigma$ for any $\mathbf{n} \in \mathbb{Z}^d$.

We denote by Φ the set of all Lyapunov weights of a. In the qualitative analysis of Lyapunov weights we shall presently conduct, the precise size of a single weight is immaterial. Instead, we shall be interested in the set of all $\alpha \in \Phi$ (and the corresponding eigenspaces) for which $\alpha(\mathbf{n}) > 0$ (or $\alpha(\mathbf{n}) < 0$), for some fixed $\mathbf{n} \in \mathbb{Z}^d$. Moreover, the subspace of all eigenvectors corresponding to a given Lyapunov weight α is not in general a subalgebra of $\mathfrak{g}$. The following definition takes care of both issues.

Definition 3.3. A *coarse Lyapunov weight* of a is an equivalence class for the equivalence relation $\sim$ on Φ defined by $\alpha \sim \beta$ if there exists $c > 0$ such that $\alpha = c\beta$.

We shall indicate with $[\Phi] = \Phi/\sim$ the set of all coarse Lyapunov weights of a.

Given $[\alpha] \in [\Phi]$ a coarse Lyapunov weight of a, define

$$\Sigma_{[\alpha]} \coloneqq \{\chi\colon \mathbb{Z}^d \to \overline{\mathbb{Q}_S} : \chi \text{ is a character of } a \text{ and } \log|\chi|_S \in [\alpha]\}$$

and

$$\mathfrak{g}^{[\alpha]} \coloneqq \sum_{\chi\in\Sigma_{[\alpha]}} \mathfrak{g}^\chi\ ;$$

we refer to $\mathfrak{g}^{[\alpha]}$ as the *coarse Lyapunov weight space* corresponding to $[\alpha]$.

Lemma 3.4 ([9, Prop. 4.9, Prop. 4.11])**.** *For any coarse Lyapunov weight $[\alpha] \in [\Phi]$ of a, the subspace $g^{[\alpha]}$ is a nilpotent subalgebra of $\mathfrak{g}$.*

Corollary 3.5 ([9, Prop. 4.11])**.** *Let $[\alpha] \in [\Phi]$ be a coarse Lyapunov weight of a. Then the exponential map* $\exp\colon \mathfrak{g}^{[\alpha]} \to G$ *is well-defined everywhere and its image $G^{[\alpha]}$ is a closed unipotent subgroup of G with Lie algebra $\mathfrak{g}^{[\alpha]}$.*

We shall actually need a slight generalization of the former construction. For an arbitrary collection $\Psi \subset \Phi$ of Lyapunov weights of a, define

$$\mathfrak{g}^{[\Psi]} = \sum_{\alpha\in\Psi} \mathfrak{g}^{[\alpha]}\ .$$

The analogue of Corollary 3.5 in this more general context reads as follows:

Lemma 3.6 ([9, Prop. 4.11])**.** *Assume $\Psi \subset \Phi$ is a collection of Lyapunov weights of a satisfying $[\Psi + \Psi] \cap [\Phi] \subset [\Psi]$. Then the subspace $\mathfrak{g}^{[\Psi]}$ is a nilpotent subalgebra of $\mathfrak{g}$. Furthermore, the associated group $G^{[\Psi]} = \exp\mathfrak{g}^{[\psi]}$ is a closed unipotent subgroup of G with Lie algebra $\mathfrak{g}^{[\Psi]}$.*

As in [17], a *Zariski-connected unipotent subgroup* of G is intended to be the image under the exponential map of a Lie subalgebra of $\mathfrak{g}^{[\Psi]}$, where $\Psi \subset \Phi$ satisfies the condition in Lemma 3.6.

We now come to the definition of stable and unstable subgroups for a fixed element of the group $a(\mathbb{Z}^d) < G$.

Definition 3.7. Let $a\colon \mathbb{Z}^d \to G$ be a diagonalizable homomorphism, Φ its associated set of Lyapunov weights, $\mathbf{n} \in \mathbb{Z}^d$. The $a(\mathbf{n})$-*stable horospherical* subgroup of G is defined as

$$G^-_{a(\mathbf{n})} \coloneqq \langle G^{[\alpha]} : \alpha \in \Phi,\ \alpha(\mathbf{n}) < 0\rangle \ .$$

Similarly, The $a(\mathbf{n})$-*unstable horospherical* subgroup of G is defined as

$$G^+_{a(\mathbf{n})} \coloneqq \langle G^{[\alpha]} : \alpha \in \Phi,\ \alpha(\mathbf{n}) > 0\rangle \ .$$

Notice that we may have equivalently set

$$G^-_{a(\mathbf{n})} = \{g \in G : a(\mathbf{n})^k g a(\mathbf{n})^{-k} \to e_G \text{ as } k \to +\infty\}$$

and

$$G^+_{a(\mathbf{n})} = \{g \in G : a(\mathbf{n})^{-k} g a(\mathbf{n})^{k} \to e_G \text{ as } k \to +\infty\} \ ,$$

which in particular gives readily $G^+_{a(\mathbf{n})} = G^-_{a(-\mathbf{n})}$ (which follows also directly from Definition 3.7).

Remark 3.8. A brief remark about terminology is in order. We will focus our attention on the action of a single element $a(\mathbf{n})$ ($\mathbf{n} \in \mathbb{Z}^d$) on the space $X = \Gamma\backslash G$, where recall that an element $g \in G$ acts on X by $g \cdot \Gamma h = \Gamma h g^{-1}$. Now suppose $x \in X$, $g \in G^-_{a(\mathbf{n})}$ and let $y = g \cdot x$; then, for every integer k,

$$a(\mathbf{n})^k \cdot y = a(\mathbf{n})^k g a(\mathbf{n})^{-k} \cdot (a(\mathbf{n})^k \cdot x) \ ,$$

so that

$$d_X(a(\mathbf{n})^k \cdot x, a(\mathbf{n})^k \cdot y) \le d_G(a(\mathbf{n})^k g a(\mathbf{n})^{-k}, e_G) \to 0 \text{ as } k \to +\infty \ .$$

Hence the orbit of x under the action of the subgroup $G^-_{a(\mathbf{n})}$ is contained in the *stable manifold* through x for the action of the element $a(\mathbf{n})$ (cf. [7]). This property is the reason why $G^-_{a(\mathbf{n})}$ is called the *stable* horospherical subgroup. The same considerations apply to $G^+_{a(\mathbf{n})}$ as well.

3.2. **The class-$\mathcal{A}'$ assumption.** The upcoming definition of class-$\mathcal{A}'$ elements and homomorphisms is taken from [17].

Definition 3.9. Let $G < \mathbf{G}(\mathbb{Q}_S)$ be an S-algebraic group. A diagonalizable element $a \in G$ is said to be *of class-$\mathcal{A}'$* if the following hold:

- the projection of a to $\mathbf{G}(\mathbb{R})$ has positive real eigenvalues;
- for each finite $p \in S$, the projection of a to $\mathbf{G}(\mathbb{Q}_p)$ is such that all of its eigenvalues are powers of λ_p, where $\lambda_p \in \mathbb{Q}_p^\times$ is some invertible element with $|\lambda_p|_p \neq 1$.

A subgroup $A < G$ is said to be of class-$\mathcal{A}'$ if every element of A is of class-$\mathcal{A}'$. Finally, if $d \ge 1$ is an integer, a homomorphism $a\colon \mathbb{Z}^d \to G$ is said to be of class-$\mathcal{A}'$ if its image $a(\mathbb{Z}^d)$ is a subgroup of G of class-$\mathcal{A}'$.

Throughout the article, we shall only consider class-$\mathcal{A}'$ subgroups arising as images of class-$\mathcal{A}'$ homomorphisms; in particular, they are always simultaneously diagonalizable.

3.3. **The assumption on the solvable factor.** In order to elucidate the meaning of the requirement we impose on the measure m_Y on the solvable quotient, we briefly recall the entropy formula for translations on homomogeneous spaces (cf. [42, Sec. 9], [36, Thm. 2.1.3]).

Proposition 3.10. *Let $Y = \Lambda\backslash B$ be an S-arithmetic quotient, $a = (a_\sigma)_{\sigma\in S} \in B$ a diagonalizable element, μ an a-invariant probability measure on Y. Let $\lambda_1, \dots, \lambda_r \in \bigcup_{\sigma\in S} \overline{\mathbb{Q}_\sigma}$ be those eigenvalues of the adjoint automorphisms $\mathrm{Ad}_{a_\sigma}\colon \mathrm{Lie}(\mathbf{B}(\mathbb{Q}_\sigma)) \to \mathrm{Lie}(\mathbf{B}(\mathbb{Q}_\sigma))$ of σ-adic absolute value strictly less than* 1*. Then the entropy with respect to μ of the transformation induced by a on Y is bounded from above by*

$$h_\mu(a) \le -\sum_{i=1}^{r} m_i \log|\lambda_i| \ , \tag{3.1}$$

where m_i is the multiplicity of the eigenvalue λ_i for any $i = 1, \dots, r$. Moreover, if Λ is a lattice in B and μ is the Haar-Siegel measure on $\Lambda\backslash B$, then equality holds.

The precise formulation of the maximal-entropy assumption we place on the measure m_Y reads thus as follows: we say that m_Y has maximal entropy with respect to the action of a diagonalizable subgroup $A_B < B$ if, for any $a \in A_B$, equality holds in the formula (3.1).

Since coarse Lyapunov subgroups for the $\mathbb{Z}^d$-action associated to the homomorphism a_B are unipotent (Corollary 3.5), they are all contained in $R_u(\mathbf{B})(\mathbb{Q}_S)$. Therefore, Proposition 5.3 readily implies that any $a_B(\mathbb{Z}^d)$-invariant measure m_Y which is additionally invariant under $R_u(\mathbf{B})(\mathbb{Q}_S)$ has maximal entropy with respect to the $a_B(\mathbb{Z}^d)$-action on Y. This provides a rich class of $\mathbb{Q}$-algebraic measures (cf. Section 1) satisfying the maximal entropy assumption, namely any $\mathbb{Q}$-algebraic measure supported on a translated orbit $\Lambda B_1 b$, for an element $b \in B$ and a finite-index subgroup $B_1 < \mathbf{B}_1(\mathbb{Q}_S)$ of the $\mathbb{Q}_S$-points of a $\mathbb{Q}$-subgroup $\mathbf{B}_1$ containing $R_u(\mathbf{B})$.

An informative example: toral automorphisms. As a salient class of actions on solvable quotients dealt with by Theorem 1.1, we mention $\mathbb{Z}^d$-actions by automorphisms of real compact nilmanifolds, which can be recast in the framework of actions by translations on homogeneous spaces. The same realization can be performed for automorphisms of compact abelian groups of the form $\mathbf{G}_a^n(\mathcal{O}_S)\backslash\mathbf{G}_a^n(\mathbb{Q}_S)$, for $n \geq 1$ an integer.

For simplicity of exposition, we discuss the case of the compact abelian group $\mathbb{T}^n = \mathbb{Z}^n\backslash\mathbb{R}^n$. It is well-known that any orientation-preserving automorphism of the Lie group $\mathbb{T}^n$ is given by $\mathbb{T}^n \ni \mathbb{Z}^n + x \mapsto \mathbb{Z}^n + bx \in \mathbb{T}^n$, where b is a matrix in $\mathrm{SL}_n(\mathbb{Z})$ and bx denotes the standard matrix product between b and the column vector $x \in \mathbb{R}^n$. Let now $b_1, \dots, b_d$ be d commuting matrices in $\mathrm{SL}_n(\mathbb{Z})$, and assume that they are diagonalizable (over $\overline{\mathbb{Q}}$); denote by $\mathbf{D}$ the Zariski closure, inside the $\mathbb{Q}$-group SL_n, of the subgroup generated by $\{b_1, \dots, b_d\}$. Then $\mathbf{D}$ is a diagonalizable $\mathbb{Q}$-group,[9] acting canonically by automorphisms of the commutative, unipotent $\mathbb{Q}$-group $\mathbf{G}_a^n$; we may thus form the semidirect[10] product $\mathbf{B} = \mathbf{D} \ltimes \mathbf{G}_a^n$, which is a solvable $\mathbb{Q}$-group with unipotent radical $\mathbf{G}_a^n$.

The action of an element $(b_0, 0) \in \mathbf{B}(\mathbb{R})$ by right translations on $\mathbf{B}(\mathbb{Z})\backslash\mathbf{B}(\mathbb{R})$ is given by $\mathbf{B}(\mathbb{Z})(b, x) \mapsto \mathbf{B}(\mathbb{Z})(bb_0, x)$, indicating with bb_0 the standard matrix product, as before.

Notice that, if b_0 is an element of $\mathbf{D}(\mathbb{Z})$ and $b = \mathbb{1}_n$ is the identity matrix, then $\mathbf{B}(\mathbb{Z})(bb_0, x) = \mathbf{B}(\mathbb{Z})(b_0, x) = \mathbf{B}(\mathbb{Z})(\mathbb{1}_n, b_0^{-1}x)$. Therefore, the projection of $\mathbf{G}_a^n(\mathbb{R}) = \mathbb{R}^n < \mathbf{B}(\mathbb{R})$ to $\mathbf{B}(\mathbb{Z})\backslash\mathbf{B}(\mathbb{R})$, which is homeomorphic to $\mathbb{T}^n$, is invariant under the $\mathbb{Z}^d$-action on $\mathbf{B}(\mathbb{Z})\backslash\mathbf{B}(\mathbb{R})$ arising from the homomorphism $a_B \colon \mathbb{Z}^d \to \mathbf{B}(\mathbb{R})$ given by $a_B(e_i) = b_i, i = 1, \dots, d$; furthermore, this restricted action is topologically conjugated to the $\mathbb{Z}^d$-action on $\mathbb{T}^n$ induced by the matrices $b_1, \dots, b_d$.

3.4. **Failure of disjointness in rank one.** Let us now explicate, by means of counterexamples, how Theorem 1.1 fails in the absence of the higher-rank assumptions on the homomorphisms a_G. Suppose that $\mathbb{Z}$ acts on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ by

$$n \cdot \mathrm{SL}_2(\mathbb{Z})g = \mathrm{SL}_2(\mathbb{Z})g \begin{pmatrix} e^{n/2} & 0 \\ 0 & e^{-n/2} \end{pmatrix}, \; g \in \mathrm{SL}_2(\mathbb{R}), n \in \mathbb{Z};$$

this is the $\mathbb{Z}$-action on the unit tangent bundle of the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ via the $\times 1$-map of the geodesic flow (cf. [21, Sec. 9.4]); as shown by Ornstein and Weiss in [49], geodesic flows on hyperbolic surfaces have the Bernoulli property, which in particular entails that the invertible measure-preserving system given by the $\times 1$ map of the flow is measure-theoretically isomorphic to a Bernoulli shift. For the solvable factor, we take the $\mathbb{Z}$-action on the two-torus $\mathbb{T}^2$ given by the hyperbolic matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; this induces what is known as an hyperbolic toral

[9] The group $\mathbf{D}$ is defined over $\mathbb{Q}$ since the matrices $b_1, \dots, b_d$ have rational coefficients, see [40, Chap. I].

[10] For the sake of clarity, the group law we consider here is given by $(d, x)(d', x') = (dd', x + dx')$ for any $(d, x), (d', x') \in \mathbf{B}$.

automorphism, which is again isomorphic to a Bernoulli shift [34]. Therefore the two $\mathbb{Z}$-actions are isomorphic, which is at the opposite extreme of being disjoint.

4. Disjointness without eigenvalue restrictions on the solvable factor

The aim of this section is to illustrate how the statement of Theorem 1.1 can be derived from the version phrased in Theorem 1.3, that is, from the case where the diagonal homomorphism $a\colon \mathbb{Z}^d \to H$ is of class-$\mathcal{A}'$. This reduction then enables us to apply Proposition 1.4, in which the class-$\mathcal{A}'$ assumption is essential (cf. [42]), to the S-arithmetic quotient $\Delta\backslash H$, as outlined in the introduction and thoroughly carried out in Section 8.

The deduction of Theorem 1.1 from Theorem 1.3 relies crucially on the decomposition of diagonalizable elements into compact and non-compact parts; for the reader's convenience, we briefly recall it hereunder (see also [17, Cor. 3.3] and its proof).

Let $\mathbf{G}$ be a connected $\mathbb{R}$-group, $a \in \mathbf{G}(\mathbb{R})$ an element which is diagonalizable over $\mathbb{C}$. There is a unique decomposition $a = a_{\mathrm{ell}}a_{\mathrm{nc}}$ into commuting semisimple elements $a_{\mathrm{ell}}, a_{\mathrm{nc}} \in \mathbf{G}(\mathbb{R})$ such that a_{ell} has only complex eigenvalues of absolute value one and a_{nc} has positive real eigenvalues. The elements a_{ell} and a_{nc} are called, respectively, the *elliptic* and *non-compact* part of a. If $(a_\lambda)_{\lambda\in\Lambda}$ is a commuting family of diagonalizable elements, then their elliptic and non-compact parts all commute with each other; moreover, there exists an $\mathbb{R}$-split subtorus $\mathbf{T} < \mathbf{G}$ such that the connected component, for the analytic topology on $\mathbf{G}(\mathbb{R})$, of the group $\mathbf{T}(\mathbb{R})$ of its real points contains all the non-compact parts of the collection $(a_\lambda)_{\lambda\in\Lambda}$.

Remark 4.1. If $G < \mathbf{G}(\mathbb{R})$ is a finite-index subgroup containing the set $\{a_\lambda\}_{\lambda\in\Lambda}$, then it contains the set of their non-compact parts (and hence the set of their compact parts as well), since G contains any connected subgroup of $\mathbf{G}(\mathbb{R})$.

Fix now a non-archimedean place $\sigma = p$, for a positive prime p, and recall that the unique extension of $|\cdot|_p$ to an absolute value on a fixed algebraic closure $\overline{\mathbb{Q}_p}$ takes values in the set $\{0\} \cup \{p^\alpha : \alpha \in \mathbb{Q}\}$. Suppose that $\mathbf{G}$ is a connected $\mathbb{Q}_p$-group, and let $a \in \mathbf{G}(\mathbb{Q}_p)$ be a diagonalizable element all of whose eigenvalues (in $\overline{\mathbb{Q}_p}$) have p-adic absolute value which is an integer power of p; this is the case, for instance, if a is diagonalizable over $\mathbb{Q}_p$. Then there is a unique decomposition $a = a_{\mathrm{ell}}a_{\mathrm{nc}}$ into commuting semisimple elements $a_{\mathrm{ell}}, a_{\mathrm{nc}} \in \mathbf{G}(\mathbb{Q}_p)$, referred to, respectively, as the elliptic and the non-compact part of a, such that all eigenvalues of a_{ell} have p-adic absolute value one, and all eigenvalues of a_{nc} are integer powers of the uniformizer $p \in \mathbb{Q}_p$. As for the real place, if we start with a commuting family of diagonalizable elements in $\mathbf{G}(\mathbb{Q}_p)$, then all their elliptic and non-compact parts commute with each other; furthermore, all the elliptic parts belong to the compact subgroup $\mathbf{G}(\mathbb{Z}_p) < \mathbf{G}(\mathbb{Q}_p)$. Finally, if $a \in \mathbf{G}(\mathbb{Q}_p)$ is diagonalizable over $\overline{\mathbb{Q}_p}$, the decomposition into elliptic and non-compact part can be applied to a power a^j for some integer $j \geq 1$.

Remark 4.2. Suppose a is a diagonalizable element contained in a subgroup $G < \mathbf{G}(\mathbb{Q}_p)$ of finite index $\ell \in \mathbb{N}$, and let $a^j = a_{\mathrm{ell}}a_{\mathrm{nc}}$ be the elliptic-non compact splitting in $\mathbf{G}(\mathbb{Q}_p)$ of a power of a. Applying Lagrange's theorem to a normal finite-index[11] subgroup $N < \mathbf{G}(\mathbb{Q}_p)$ contained in G, it follows that the powers $a_{\mathrm{ell}}^{\ell!}$ and $a_{\mathrm{nc}}^{\ell!}$ both belong to N, and hence to G.

If now $a_1, \ldots, a_d$ are d commuting diagonalizable elements contained in G and generating a subgroup A, let j be an integer such that a_i^j admits an elliptic-non compact decomposition for every $i = 1, \ldots, d$. Then the elliptic and non-compact parts of $a_1^{j\ell!}, \ldots, a_d^{j\ell!}$ belong to G, and the subgroup A_V generated by $a_1^{j\ell!}, \ldots, a_d^{j\ell!}$ has finite index in A.

[11]Such a normal subgroup N can be obtained as the intersection of all conjugates of G inside $\mathbf{G}(\mathbb{Q}_p)$; its index in $\mathbf{G}(\mathbb{Q}_p)$ is at most $\ell!$.

A further essential ingredient in the reduction is the ergodic decomposition of invariant measures with respect to actions of subgroups. The upcoming section serves the purpose of recalling the related formalism and the relevant results.

4.1. **Conditional measures and ergodic decomposition.** The ergodic decomposition of invariant measures for group actions lends itself to a description in terms of conditional measures with respect to an appropriate sub-σ-algebra. The language of conditional measures will be also employed in Section 5 as a convenient framework in which to inscribe the more refined notion of leafwise measures. We thus begin by setting up basic notation concerning conditional measures. The main references for this paragraph are [21, Chap. 5], [63] and [68].

Let X be a set, $\mathcal{B}$ a σ-algebra of subsets of X, such that the pair $(X, \mathcal{B})$ is a standard Borel space, that is, there exists a measurable isomorphism from $(X, \mathcal{B})$ to the Borel space $(Y, \mathcal{B}_Y)$ associated to a compact metrizable space Y. We denote by $\mathcal{M}^+(X)$ the set of positive finite measures on $(X, \mathcal{B})$, endowed with the coarsest σ-algebra for which the maps $\nu \mapsto \nu(B), B \in \mathcal{B}$ are measurable. If Z is a set and $\mathcal{C}$ is a σ-algebra of subsets of Z, a *$\mathcal{C}$-measurable measure-valued function* on Z is meant to be a function $Z \to \mathcal{M}^+(X)$ which is measurable when $\mathcal{M}^+(X)$ is endowed with the σ-algebra just described, and Z is equipped with σ-algebra $\mathcal{C}$.

For any $\nu \in \mathcal{M}^+(X)$, we denote by $\mathcal{L}^1(X, \mu)$ the vector space of complex, ν-integrable functions defined on X (here we do *not* identify functions that agree ν-almost everywhere).

Definition 4.3. Let $X, \mathcal{B}, \mathcal{M}^+(X)$ be as above, and let μ be probability measure on $(X, \mathcal{B})$. If $\mathcal{A} \subset \mathcal{B}$ is a sub-σ-algebra, a family of *conditional measures of μ given $\mathcal{A}$* is defined as a collection $\{\mu_x^{\mathcal{A}}\}_{x \in X}$ of probability measures on $(X, \mathcal{B})$ satisfying the following two conditions:

(1) the assignment $X \ni x \mapsto \mu_x^{\mathcal{A}} \in \mathcal{M}^+(X)$ defines an $\mathcal{A}$-measurable measure-valued function;
(2) for any $f \in \mathcal{L}^1(X, \mu)$, the $\mathcal{A}$-measurable function
$$X \ni x \mapsto \int_X f \, d\mu_x^{\mathcal{A}}$$
is a conditional expectation of f given $\mathcal{A}$.

Phrased more accurately, the second property amounts to the assertion that, for any $f \in \mathcal{L}^1(X, \mu)$ and any measurable set $A \in \mathcal{A}$,
$$\int_A f \, d\mu = \int_A \left(\int_X f(y) d\mu_x^{\mathcal{A}}(y) \right) d\mu(x) \, .$$
Heuristically, the function $x \mapsto \int_X f d\mu_x^{\mathcal{A}}$ represents the best approximation of f given the knowledge of the events in the σ-algebra $\mathcal{A}$.

Conditional measures always exist in our current setup (for a detailed proof of this fact, we refer the reader to [21, Thm. 5.14] or alternatively to [63, Sec. I.3.5]). A σ-algebra $\mathcal{A}$ is *countably generated* if there exists a subset $S \subset \mathcal{A}$ which is at most countable and generates $\mathcal{A}$. In this case, the *atom* $[x]_{\mathcal{A}}$ of a point $x \in X$ with respect to $\mathcal{A}$, defined as the intersection of all $A \in \mathcal{A}$ containing x, is itself an element of $\mathcal{A}$; moreover, it turns out that each conditional measure $\mu_x^{\mathcal{A}}$ is concentrated on the atom $[x]_{\mathcal{A}}$, at least for μ-almost every point $x \in X$. In this sense, we may loosely say that the conditional measure $\mu_x^{\mathcal{A}}$ describes μ on the $\mathcal{A}$-atom of x.

We now turn to the formal notion of ergodic decomposition in the context of group actions. Let G be a locally compact group, acting measurably[12] on a standard Borel space $(X, \mathcal{B})$. Let $\mathcal{M}^1(X) \subset \mathcal{M}^+(X)$ denote the subset of probability measures on $(X, \mathcal{B})$, endowed with the induced measurable structure from $\mathcal{M}^+(X)$. Suppose that $\mu \in \mathcal{M}^1(X)$ is invariant under the

[12] A group action of a topological group G on a measurable space $(X, \mathcal{B})$ is called measurable if the action map $G \times X \to X$ is measurable, when $G \times X$ is endowed with the product σ-algebra $\mathcal{B}_G \otimes \mathcal{B}$.

G-action. A *G-ergodic decomposition* of μ is a measurable assignment $(Y, \mathcal{C}, \rho) \ni y \mapsto \mu_y \in \mathcal{M}^1(X)$, where $(Y, \mathcal{C}, \rho)$ is a probability measure space, such that:

(1) for ρ-almost every $y \in Y$, μ_y is invariant and ergodic with respect to the G-action;
(2) it holds that

$$\int_X f \, d\mu = \int_Y \int_X f(x) \mathrm{d}\mu_y(x) \, \mathrm{d}\rho(y)$$

for every $f\colon X \to \mathbb{C}$ bounded and measurable.

Concisely, we shall say that $\mu = \int_Y \mu_y \, \mathrm{d}\rho(y)$ is a G-ergodic decomposition of μ.

A G-ergodic decomposition $\mu = \int_Y \mu_y \, \mathrm{d}\rho(y)$ induces, via push-forward of the measure ρ on Y, a probability measure on the measurable space $\mathcal{M}^1(X)$.

The following result is paramount in the ergodic theory of group actions, giving existence and uniqueness of the ergodic decomposition of an invariant measure.

Theorem 4.4. *Let G be a locally compact group acting measurably on a standard Borel space $(X, \mathcal{B})$, μ a G-invariant probability measure on $(X, \mathcal{B})$. There exists a G-ergodic decomposition of μ. Moreover, if $\mu = \int_Y \mu_y \mathrm{d}\rho_Y(y)$ and $\mu = \int_Z \mu_z \mathrm{d}\rho_Z(z)$ are two such decompositions, then the probability measures induced on $\mathcal{M}^1(X)$ by ρ_Y and ρ_Z coincide.*

For the existence part of the theorem, we refer the reader to [27,68]; uniqueness of the ergodic decomposition is shown, for instance, in [52, Sec. 12].

Remark 4.5. Measurable actions of locally compact groups on standard Borel spaces always admit *topological models*. More precisely, given a measurable action of a locally compact group G on a standard Borel space $(X, \mathcal{B})$, there is a compact metrizable space Y, a continuous action of G on Y, and a G-equivariant measurable isomorphism $\phi\colon (X, \mathcal{B}) \to (Y, \mathcal{B}_Y)$; this is also shown in [68]. We shall appeal to this result in the proof of Proposition 8.7.

Ergodic decompositions relate to conditional measures in the following way: in the setting of Theorem 4.4, let $\mathcal{E} \subset \mathcal{B}$ be the σ-algebra of G-invariant measurable sets on X. Let $\{\mu_x^{\mathcal{E}}\}_{x \in X}$ be a family of conditional measures of μ given $\mathcal{E}$. Then the assignment $(X, \mathcal{B}, \mu) \mapsto \mu_x^{\mathcal{E}} \in \mathcal{M}^1(X)$ defines a G-ergodic decomposition of μ.

Let us now consider the following situation, which will frequently present itself in the sequel. Suppose that μ is invariant and ergodic under the action of a locally compact group G, and let $G' < G$ be a closed, normal subgroup such that the quotient group G'/G is compact and abelian. In this case, the G'-ergodic decomposition of μ takes a particularly simple form. Let $m_{G/G'}$ be the unique probability Haar measure on the compact group G/G', and fix a G'-ergodic decomposition $(Y, \mathcal{B}, \rho) \ni y \mapsto \mu_y$ of μ. Choose an *ergodic component* ν of μ, that is a G'-ergodic measure belonging to the topological support of the probability measure induced by ρ on $\mathcal{M}^1(X)$. The push-forward measure $g_*\nu$ under the action of an element $g \in G$ depends only on the left coset gG', as ν is G'-invariant. The quotient G/G' being abelian, the measure $g_*\nu$ is G'-invariant and ergodic for any $g \in G$. More is true, namely:

Proposition 4.6. *Suppose ν is a G'-invariant and ergodic probability measure on X. Then the measure*

$$\mu = \int_{G/G'} g_*\nu \, \mathrm{d}m_{G/G'}(gG')$$

is G-invariant and ergodic.

Conversely, if μ is G-invariant and ergodic, and ν is a G'-ergodic component of μ, then

$$\mu = \int_{G/G'} g_*\nu \, \mathrm{d}m_{G/G'}(gG')$$

is a G'-ergodic decomposition of μ.

4.2. **Proof of Theorem 1.1 assuming Theorem 1.3.** We are now in a position to deduce Theorem 1.1, assuming the validity of Theorem 1.3. Therefore, suppose $a\colon \mathbb{Z}^d \to H$ is the product of two diagonalizable homomorphisms $a_G\colon \mathbb{Z}^d \to G$ and $a_B\colon \mathbb{Z}^d \to B$ satisfying the hypothesis in Theorem 1.1, and let μ be an ergodic joining of the resulting measure-preserving $\mathbb{Z}^d$-actions on (X, m_X) and (Y, m_Y). Combining the decomposition into elliptic and non-compact parts for the different places $\sigma \in S$ (cf. the beginning of Section 4, in particular Remarks 4.1 and 4.2), we infer the existence of two group homomorphisms $a_{\mathrm{ell}}, a_{\mathrm{nc}}\colon \mathbb{Z}^d \to H$ and a finite-index subgroup A_V of $A = a(\mathbb{Z}^d)$ satisfying the following properties:

(1) $A_V = \{a_{\mathrm{nc}}(\mathbf{n})a_{\mathrm{ell}}(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^d\}$;
(2) $a_{\mathrm{nc}}(\mathbf{n}), a_{\mathrm{ell}}(\mathbf{m})$ commute for any $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^d$;
(3) the image of a_{ell} is relatively compact in H;
(4) the image A_{nc} of a_{nc} is a subgroup of class-$\mathcal{A}'$.

Remark 4.7. Observe that, if $H = \mathbf{H}(\mathbb{Q}_S)$ and the elements of A are diagonalizable over $\mathbb{Q}_S$, then A_V can be taken to be equal to A.

We aim to manufacture an A_{nc}-invariant and ergodic measure, to which Theorem 1.3 applies. Denote by M the closure of $a_{\mathrm{ell}}(\mathbb{Z}^d)$ with respect to the analytic topology on H; it is a compact abelian group contained in the centralizer $C_G(A_{\mathrm{nc}})$ of A_{nc}. Choose an A_V-ergodic component μ_0 of μ; since A/A_V is a finite abelian group, Proposition 4.6 gives that

$$\mu = \frac{1}{[A : A_V]} \sum_{a' A_V \in A/A_V} a'_* \mu_0 \tag{4.1}$$

is an A_V-ergodic decomposition of μ. Projecting (4.1) to X, we deduce that

$$m_X = \frac{1}{[A : A_V]} \sum_{a' A_V \in A/A_V} (\pi_G(a'))_* (\pi_X)_* \mu_0$$

is a $\pi_G(A_V)$-ergodic decomposition of m_X, where $\pi_G\colon H \to G$ and $\pi_X\colon X \times Y \to X$ denote the canonical projection maps. As finite-index subgroups of $\mathbb{Z}^d$ act ergodically on (X, m_X) (see Remark 1.2), it follows that $m_X = (\pi_X)_*\mu_0$, by uniqueness of the ergodic decomposition. Similarly, we deduce that μ_0 projects to m_Y on Y.

Let now A' be the subgroup generated by A_V and M, which coincides with the product set $A_V M$ since A_V and M commute. Notice that A' also coincides with the group $A_{\mathrm{nc}}M$. Endow A' with the final topology derived from the surjective product morphism $A_V \times M \ni (a, g) \mapsto ag \in A'$, where A_V is equipped with the discrete topology and M with the topology induced from H; such a topology makes A' into a locally compact second countable topological group. The subgroup $A_V < A'$ is discrete, hence closed, and the quotient A'/A_V is isomorphic, as a topological group, to the quotient $M/(A_V \cap M)$, the latter being a compact abelian group. By virtue of Proposition 4.6, the measure

$$\mu' := \int_{A'/A_V} a'_*\mu_0 \, \mathrm{d}m_{A'/A_V}(a' A_V) \tag{4.2}$$

is A'-invariant and ergodic.

At this point, we endow A' with a possibly different topology, namely the final topology coming from the product morphism $A_{\mathrm{nc}} \times M \to A'$, where A_{nc} is equipped with the discrete topology. In this way A_{nc} becomes a discrete, hence closed subgroup of A', and the quotient A'/A_{nc} is isomorphic to $M/(A_{\mathrm{nc}} \cap M)$, that is to a compact abelian group. If μ_{nc} is an A_{nc}-ergodic component of μ', we may invoke Proposition 4.6 once more to infer that

$$\mu' = \int_{A'/A_{\mathrm{nc}}} a'_*\mu_{\mathrm{nc}} \, \mathrm{d}m_{A'/A_{\mathrm{nc}}}(a' A_{\mathrm{nc}}) \tag{4.3}$$

is an A_{nc}-ergodic decomposition of μ'.

In order to apply Theorem 1.3 to μ_{nc}, we need to identify its projections onto X and Y. Since μ_0 projects to the Haar measure m_X on X, the same holds true for μ' by (4.2). Projecting (4.3) to X, it follows that

$$m_X = \int_{A'/A_{\mathrm{nc}}} \pi_G(a')_*(\pi_X)_*\mu_{\mathrm{nc}} \,\mathrm{d}m_{A'/A_{\mathrm{nc}}}(a'A_{\mathrm{nc}})$$

is a $\pi_G(A_{\mathrm{nc}})$-ergodic decomposition of m_X. We now appeal to:

Lemma 4.8. *The Haar measure m_X is ergodic under the action of the subgroup $\pi_G(A_{\mathrm{nc}})$.*

For the proof of this lemma, we rely on the weak-mixing[13] properties of actions of certain class-$\mathcal{A}'$ elements on perfect quotients saturated by unipotents, which are established in [17]. Specifically, we shall make use of the following result:

Proposition 4.9 ([17, Prop. 3.1])**.** *Let $X = \Gamma\backslash G$ be an S-arithmetic quotient of a Zariski-connected perfect $\mathbb{Q}$-group $\mathbf{G}$. Let m_X be the Haar measure on X, and assume that X is saturated by unipotents. Let $a \in G$ be an element of class-$\mathcal{A}'$ whose projection onto the $\mathbb{Q}_S$-points of any $\mathbb{Q}$-almost simple factor of $\mathbf{G}$ is non-trivial. Then the action of a on X is weak mixing with respect to m_X.*

Proof of Lemma 4.8. Recall our topological assumption on the homomorphism $a_G\colon \mathbb{Z}^d \to G$: its projection onto the $\mathbb{Q}_S$-points of every $\mathbb{Q}$-almost simple factor of $\mathbf{G}$ is proper. This carries over to the restriction of a_G to the finite-index subgroup $a^{-1}(A_V) < \mathbb{Z}^d$. Since $a_G(a^{-1}(A_V))$ and $\pi_G(A_{\mathrm{nc}})$ differ only by a compact group, the same property is also enjoyed by the homomorphism $\pi_G \circ a_{\mathrm{nc}}\colon a^{-1}(A_V) \to \pi_G(A_{\mathrm{nc}})$. Denote by $\pi_i, i \in I$, the various projections of the latter homomorphism onto the $\mathbb{Q}_S$-points of the $\mathbb{Q}$-almost simple factors of $\mathbf{G}$, where I is a finite set. As the non-compact subgroup A_V cannot be the finite union of the compact sets $\pi_i^{-1}(e)$, e being the identity element of the corresponding factor, it follows that there exists an element $a_0 \in \pi_G(A_{nc})$ whose projection under π_i is non-trivial for all $i \in I$. Since a_0 is a class-$\mathcal{A}'$ element by definition of the group A_{nc}, Proposition 4.9 yields that the action of the cyclic group generated by a_0 on X is weak mixing, hence in particular ergodic, with respect to m_X. A fortiori, m_X is ergodic under the action of the larger group $\pi_G(A_{\mathrm{nc}})$. □

Lemma 4.8, coupled with uniqueness of the ergodic decomposition, delivers $(\pi_X)_*\mu_{\mathrm{nc}} = m_X$.

We now examine the projection of μ_{nc} onto the solvable factor; for notational convenience, we denote it by $m_{Y,\mathrm{nc}}$. First, we claim that m'_Y has maximal entropy for the action of the group $\pi_B(A_{\mathrm{nc}})$. For this, we fundamentally rely on the relationship between maximality of entropy and invariance under coarse Lyapunov subgroups expressed in Proposition 5.3. Our assumption that m_Y has maximal entropy with respect to the action of $a_B(\mathbb{Z}^d)$ (cf. Proposition 3.10) implies, applying Propositions 5.3 and 5.4 to the action of elements of $a_B(\mathbb{Z}^d)$, that μ is invariant under all coarse Lyapunov subgroups associated to non-trivial coarse Lyapunov weights (cf. Section 3.1) for the $a_B(\mathbb{Z}^d)$-action on Y. This invariance property transfers at once to the measure m'_Y, projecting (4.2) to Y and noticing that $\pi_B(M)$ normalizes the subgroup of B generated by all such coarse Lyapunov subgroups. It now suffices to observe that, by the very nature of the elliptic-non compact decomposition, the actions of the groups $a_B(\mathbb{Z}^d)$ and $\pi_B(A_{\mathrm{nc}})$ on Y give rise to the same collection of coarse Lyapunov subgroups; therefore,

[13] Recall that a measure-preserving transformation T on a probability measure space $(Z, \mathcal{C}, \nu)$ is said to be *weak mixing* if

$$\frac{1}{N}\sum_{n=0}^{N-1} |\nu(T^{-n}(A) \cap B) - \nu(A)\nu(B)| \overset{N\to\infty}{\longrightarrow} 0$$

for all measurable subsets $A, B \subset Z$.

appealing to Propositions 5.3 and 5.4 once more, but this time for the action of $\pi_B(A_{\mathrm{nc}})$ on Y, we achieve the proof of our claim that m'_Y has maximal entropy with respect to the latter group.

Projecting (4.3) onto Y gives

$$m'_Y = \int_{A'/A_{\mathrm{nc}}} \pi_B(a')_*(m_{Y,\mathrm{nc}}) \, \mathrm{d}m_{A'/A_{\mathrm{nc}}}(a'A_{\mathrm{nc}}) \, , \tag{4.4}$$

where m'_Y is the projection of μ' to Y. If a_0 is an element of $\pi_B(A_{\mathrm{nc}})$, then this integral representation carries over to the entropy with respect to a_0, that is,

$$h_{m'_Y}(a_0) = \int_{A'/A_{\mathrm{nc}}} h_{\pi_B(a')_*(m_{Y,\mathrm{nc}})}(a_0) \, \mathrm{d}m_{A'/A_{\mathrm{nc}}}(a'A_{\mathrm{nc}}) \, ; \tag{4.5}$$

we refer the reader to [25, Thm. 15.12] for a proof of this general relationship. As m'_Y has maximal entropy for the $\pi_B(A_{\mathrm{nc}})$-action, (4.5) forces

$$h_{m'_Y}(a_0) = h_{\pi_B(a')_*(m_{Y,\mathrm{nc}})}(a_0) \quad \text{for } m_{A'/A_{\mathrm{nc}}}\text{-almost every } a'A_{\mathrm{nc}} \in A'/A_{\mathrm{nc}},$$

since A' being abelian implies that $\pi_B(a')_*(m_{Y,\mathrm{nc}})$ is $\pi_B(A_{\mathrm{nc}})$-invariant for every $a' \in A'$. Upon replacing μ_{nc} with a translate of it by an element of A', we might therefore assume that $h_{m'_Y}(a_0) = h_{m_{Y,\mathrm{nc}}}(a_0)$. Hence, we have proved that $m_{Y,\mathrm{nc}}$ has maximal entropy with respect to the action of $\pi_B(A_{\mathrm{nc}})$.

We are now finally in a position to apply Theorem 1.3 to the measure μ_{nc}, and deduce that $\mu_{\mathrm{nc}} = m_X \times m_{Y,\mathrm{nc}}$. As a consequence,

$$\begin{aligned}
\mu' &= \int_{A'/A_{\mathrm{nc}}} a'_*\mu_{\mathrm{nc}} \, \mathrm{d}m_{A'/A_{nc}}(a'A_{\mathrm{nc}}) = \int_{A'/A_{\mathrm{nc}}} a'_*(m_X \times m_{Y,\mathrm{nc}}) \, \mathrm{d}m_{A'/A_{\mathrm{nc}}}(a'A_{\mathrm{nc}}) \\
&= \int_{A'/A_{\mathrm{nc}}} m_X \times \pi_B(a')_*(m_{Y,\mathrm{nc}}) \, \mathrm{d}m_{A'/A_{\mathrm{nc}}}(a'A_{\mathrm{nc}}) \\
&= m_X \times \int_{A'/A_{\mathrm{nc}}} \pi_B(a')_*(m_{Y,\mathrm{nc}}) \, \mathrm{d}m_{A'/A_{\mathrm{nc}}}(a'A_{\mathrm{nc}}) = m_X \times m'_Y \, ,
\end{aligned}$$

where the last equality comes from (4.4).

Recall that μ' is an average of μ_0 over the compact group M. We new set out to deduce that $\mu_0 = m_X \times m_Y$ from the product structure of μ', which has just been established.

Let $\mathbb{Z}^d$ act on the product space $(X \times Y, \mu_{\mathrm{nc}}) \times (M, m_M)$ via

$$\mathbf{n} \cdot (z, g) = \big(a_{\mathrm{nc}}(\mathbf{n}) \cdot z, a_{\mathrm{ell}}(\mathbf{n})g\big) \, , \text{ for any } z \in X \times Y, \; g \in M, \; \mathbf{n} \in \mathbb{Z}^d.$$

We let

$$m_{Y,\mathrm{nc}} \times m_M = \int_{Y \times M} \nu_{(y,g)} \, \mathrm{d}(m_{Y,\mathrm{nc}} \times m_M)(y, g)$$

be an ergodic decomposition of $m_{Y,\mathrm{nc}} \times m_M$ with respect to the $\mathbb{Z}^d$-action. Then $\tilde{\nu}_{(y,g)} = m_X \times \nu_{(y,g)}$ is a $\mathbb{Z}^d$-invariant and ergodic measure on $X \times Y \times M$, as the action of A_{nc} on (X, m_X) is mixing[14]. Let $\psi \colon X \times Y \times M \to X \times Y$ denote the action map $(z, g) \mapsto g \cdot z$. Similarly, let $\psi_Y \colon Y \times M \to Y$ denote the action map $(y, g) \mapsto g \cdot y$. If $f_1 \in \mathrm{C_c}(X)$ and $f_2 \in \mathrm{C_c}(Y)$ are compactly supported continuous functions, then

$$\begin{aligned}
\int_{X \times Y} f_1 \otimes f_2 \, \mathrm{d}\psi_* \tilde{\nu}_{(y,m)} &= \int_{Y \times M} \left(\int_X f_1(\overline{g} \cdot x) \, \mathrm{d}x \right) f_2(\overline{g} \cdot \overline{y}) \, \mathrm{d}\nu_{(y,g)}(\overline{y}, \overline{g}) \\
&= \int_X f_1 \, \mathrm{d}m_X \int_Y f_2 \, \mathrm{d}(\psi_{Y*}\nu_{(y,g)}) \, .
\end{aligned}$$

[14] Here we are invoking the well-known fact that the product of a mixing and an ergodic action is ergodic.

Density of the subspace spanned by functions of the form $f_1 \otimes f_2$ inside $\mathrm{C}_0(X \times Y)$ implies that $\psi_* \tilde{\nu}_{(y,g)} = m_X \times \psi_{Y*} \nu_{(y,g)}$. If now $z \in X \times Y$, $g \in M$, and $\mathbf{n} \in \mathbb{Z}^d$, we have

$$\psi\big(\mathbf{n} \cdot (z,g)\big) = \psi\big(a_{\mathrm{nc}}(\mathbf{n}) \cdot z, a_{\mathrm{ell}}(\mathbf{n}) g\big) = a_{\mathrm{ell}}(\mathbf{n}) g a_{\mathrm{nc}}(\mathbf{n}) \cdot z = a(\mathbf{n}) \cdot \psi(z,g) \ ,$$

whence ψ is $\mathbb{Z}^d$-equivariant, where $\mathbb{Z}^d$ acts on $X \times Y$ via the homomorphism $\mathbf{n} \mapsto a_{\mathrm{ell}}(\mathbf{n}) a_{\mathrm{nc}}(\mathbf{n})$. As ψ is $\mathbb{Z}^d$-equivariant, we know that $\psi_* \tilde{\nu}_{(y,g)}$ is $\mathbb{Z}^d$-invariant and ergodic. On the other hand, we have that, for any $f \in \mathrm{C_c}(X \times Y)$,

$$\begin{aligned}
\int_{X \times Y} f \,\mathrm{d}\mu' &= \int_M \int_{X \times Y} f(g \cdot z) \,\mathrm{d}\mu_{\mathrm{nc}}(z) \,\mathrm{d}m_M(g) \\
&= \int_{X \times Y} f \,\mathrm{d}\psi_*(\mu_{\mathrm{nc}} \times m_M) \\
&= \int_{X \times Y \times M} f \circ \psi(x,y,g) \,\mathrm{d}(m_X \times m_{Y,\mathrm{nc}} \times m_M)(x,y,g) \\
&= \int_{Y \times M} \int_{X \times Y \times M} f \circ \psi(x, \overline{y}, \overline{g}) \,\mathrm{d}\tilde{\nu}_{(y,g)}(x, \overline{y}, \overline{g}) \,\mathrm{d}(m_{Y,\mathrm{nc}} \times m_M)(y,g) \\
&= \int_{Y \times M} \int_{X \times Y} f(x, \overline{y}) \,\mathrm{d}(\psi_* \tilde{\nu}_{(y,g)})(\overline{y}) \,\mathrm{d}(m_{Y,\mathrm{nc}} \times m_M)(y,g)
\end{aligned}$$

As a result,

$$\mu' = \int_{Y \times M} \psi_* \tilde{\nu}_{(y,g)} \,\mathrm{d}(m_{Y,\mathrm{nc}} \times m_M)(y,g)$$

is an ergodic decomposition of μ' for the $\mathbb{Z}^d$-action induced by the homomorphism $\mathbf{n} \mapsto a_{\mathrm{ell}}(\mathbf{n}) a_{\mathrm{nc}}(\mathbf{n})$. In particular, by uniqueness of the ergodic decomposition, there are $g_1, g_2 \in M$ and $y \in Y$ such that

$$\mu^{\mathcal{E}} = g_1 * \tilde{\nu}_{(y,g_2)} = m_X \times g_{1*} \nu_{(y,g_2)}.$$

As $\mu^{\mathcal{E}}$ is a joining, it follows that $g_{1*} \nu_{(y,g_2)} = m_Y$ and therefore $\mu^{\mathcal{E}} = m_X \times m_Y$. As $\mu^{\mathcal{E}}$ was an arbitrary ergodic component, we finally obtain that $\mu = m_X \times m_Y$.

The reduction of Theorem 1.1 to Theorem 1.3 is thus achieved.

5. Leafwise measures and entropy contribution

In order to obtain additional invariance of the measure under a unipotent subgroup in the proof of Theorem 1.1, we rely heavily on a combination of the high and low entropy method (cf. [8–10, 14, 16]),as already mentioned in the introduction. Both tools are intended as a way to establish invariance of a certain measure μ under the action of a subgroup $H < G$ on a quotient space of the form $\Gamma \backslash G$, under appropriate positive entropy assumptions for the given action; their effectiveness in producing invariance hinges upon the relationship between entropy and *leafwise measures*, which describe μ along H-orbits in a way which is reminiscent of the description of a measure on atoms of a sub-σ-algebra provided by conditional measures (cf. Section 4.1). The main purpose of this section is therefore twofold: we first define the notion of leafwise measures in a fairly general context of continuous group actions and list a few properties which will be relevant in the sequel. Afterwards, we specialise the treatment to the case of algebraic actions on homogeneous spaces, and relate the entropy of a single acting element with respect to the measure μ to volume-growth properties of the corresponding leafwise measures. This will in turn enable us to detect invariance of the measures under suitable horospherical subgroups.

5.1. **Leafwise measures.** To state the defining properties of leafwise measures conveniently, we shall rely on the theory of conditional measures set forth in Section 4.1.

Let H be a locally compact group acting continuously on a locally compact topological space X, endowed with a positive Radon measure μ. We shall need in addition the technical condition that the orbital map $H \ni h \mapsto h \cdot x$ is injective for μ-almost every $x \in X$. We wish to identify a collection of measures on the acting group H which describe μ along the H-orbits. Tipically (and chiefly in the context of actions on homogeneous spaces), there is no countably generated sub-σ-algebra $\mathcal{A}$ whose atoms correspond to orbits for the H action; also, we are not assuming that μ is a probability measure. Hence conditional measures do not provide us with a solution rightaway.

Definition 5.1 furnishes an adequate replacement of conditional measures in the present setting. We say that a countably generated sub-σ algebra $\mathcal{A}$ of the Borel σ-algebra on X is H-*subordinate* if, for each $x \in X$, there exists an open, relatively compact subset $V_x \subset H$ such that the $\mathcal{A}$-atom of x is given by $[x]_{\mathcal{A}} = V_x \cdot x = \{h \cdot x : h \in V_x\}$. Such a set V_x will be referred to as the *shape* of the atom in what follows.

Definition 5.1. Let H, X, μ be as above. A collection $(\mu_x^H)_{x \in X'}$ of positive Radon measures on H, indexed by a μ-conull set $X' \subset X$, is called a family of *leafwise measures* if it satisfies the following property: for every Borel set $Y \subset X$ with $0 < \mu(Y) < \infty$, and for every countably generated, H-subordinate σ-algebra $\mathcal{A} \subset \mathcal{B}_Y$ on Y, a family of conditional measures $\mu_y^{\mathcal{A}}, y \in Y$ for the normalized measure $\mu|_Y/\mu(Y)$ is given by

$$\mu_y^{\mathcal{A}} = \frac{1}{\mu_y^H(V_y)} \big((\phi_y)_* (\mu_y^H|_{V_y}) \big) \, ,$$

for μ-almost every $y \in Y$, where V_y is the shape of the atom $[y]_{\mathcal{A}}$ and $\phi_y \colon H \to X$ is the orbital map defined by y.

We say that a Borel subset $X' \subset X$ is μ-conull if $\mu(X \setminus X') = 0$. Further, for any Borel subset $Y \subset X$, we define the restricted measure $\mu|_Y$ on Y by $\mu|_Y(A) = \mu(A)$ for any Borel set $A \subset Y$. The notation $\mathcal{B}_Y$ stands for the Borel σ-algebra on Y.

A construction of leafwise measures in the general setting outlined above is presented in [38] and, with minor adjustmens, in [16]. From now until the end of this section, we shall place ourselves in the following specific situation: $G < \mathbf{G}(\mathbb{Q}_S)$ is an S-algebraic group, $\Gamma < G$ an S-arithmetic subgroup, $X = \Gamma \backslash G$ the corresponding S-arithmetic quotient. Further, we let $A < G$ be a subgroup of class-$\mathcal{A}'$, and we fix an element $a \in A \setminus \{e_G\}$. We assume that μ is an A-invariant probability measure on X. Finally, the role of the group H in Def. 5.1 will be played by an a-normalized, Zariski connected algebraic subgroup $U < G_a^-$, where $G_a^- < G$ is the stable horospherical subgroup defined by a (cf. Section 3.1). Clearly, we are considering the action of H on X by right multiplication.

5.2. **Entropy contribution for horospherical subgroups.** In this paragraph, we examine the relationship between the entropy for the action of a fixed element $a \in A \setminus \{e_G\}$ with respect to an a-invariant probability measure μ and the corresponding leafwise measures μ_x^U for the subgroup U; what follows is taken entirely from [16, Sec. 7] and, yet more closely, from [17, Sec. 5], to which we refer for details and proofs of the upcoming results.

Denote by θ_a the automorphism of U defined via conjugation by a. A function $f \colon X \to \mathbb{R}$ is called *essentially a-invariant* if it coincides μ-almost everywhere with the function $f \circ T_a$, where $T_a(\Gamma g) = \Gamma g a^{-1}$ for all $\Gamma g \in X$.

Lemma 5.2 (cf. [9, Lem. 9.1]). *Let $V \subset U$ be a bounded neighborhood of the identity. For μ-almost every $x \in X$, the limit*

$$h_\mu(a, U, x) = \lim_{n\to\infty} -\frac{\log \mu_x^U(\theta_a^n(V))}{n}$$

exists and does not depend on the choice of V. The assignment $x \mapsto h_\mu(a, U, x)$ defines an essentially a-invariant, positive function on (a μ-conull subset of) X. Furthermore, we have:

(1) for μ-almost every $x \in X$,

$$h_\mu(a, U, x) \le h_{\mu_x^{\mathcal{E}}}(T_a) ,$$

where $\mu_x^{\mathcal{E}}, x \in X$ is an ergodic decomposition of μ;

(2) if μ_x^U has support inside a Zariski closed subgroup $P_x \le U$ which is normalized by a, then

$$h_\mu(a, U, x) \le \mathrm{mod}(a, P_x) ,$$

where $\mathrm{mod}(a, P_x)$ is the negative logarithm of the module of the automorphism $\theta_a|_{P_x}$.

The average

$$h_\mu(a, U) := \int_X h_\mu(a, U, x)\mathrm{d}\mu(x)$$

will be referred to as the *entropy contribution of U*.

We record here a refinement of the second assertion of Lemma 5.2, which is instrumental in several steps of our argumentation (see, for instance, Section 4.2 and Lemma 8.5) and can be seen as a generalization of Proposition 3.10 to every a-normalized subgroup of G_a^-.

Proposition 5.3 ([16, Thm. 7.9]). *Let $U < G_a^-$ be an a-normalized closed subgroup, for some $a \in A$, and let μ be an a-invariant probability measure on $X = \Gamma\backslash G$. Then the entropy contribution of U is bounded by*

$$h_\mu(a, U) \le \mathrm{mod}(a, U) ,$$

with equality holding if and only if μ is U-invariant.

Lastly, we recall the relationship between the Kolmogorov-Sinai entropy of the transformation induced by an element a on X and the entropy contribution of the horospherical subgroup G_a^-.

Proposition 5.4 ([9, Prop. 9.4]). *Let μ be an A-invariant probability measure on $X = \Gamma\backslash G$. Then $h_\mu(a) = h_\mu(a, G_a^-)$ for any $a \in A$.*

Let now $A = a(\mathbb{Z}^d)$, where $a\colon \mathbb{Z}^d \to G$ is a class-$\mathcal{A}'$ homomorphism. As in section 3.1, we denote by Φ the set of Lyapunov weights for a, and by $[\Phi]$ the set of coarse Lyapunov weights for a. Recall also that, for any $[\alpha] \in [\Phi]$, $G^{[\alpha]}$ denotes the coarse Lyapunov subgroup corresponding to $[\alpha]$, namely the unipotent subgroup whose Lie algebra is $\mathfrak{g}^{[\alpha]} = \bigoplus_{\alpha' \in [\alpha]} \mathfrak{g}^{\alpha'}$. Fix again a non-trivial element $a_0 \in A$, and denote by Ψ_{a_0} the set of Lyapunov weights whose associated coarse Lyapunov subgroups are contained in the stable horospherical subgroup $G_{a_0}^-$; formally,

$$\Psi_{a_0} = \{\alpha \in \Phi : G^{[\alpha]} < G_{a_0}^-\} .$$

One of the cornerstones of the use of leafwise measures in the work on higher-rank rigidity by Einsiedler and Katok [9], Einsiedler and Lindenstrauss [17] as well as Einsiedler, Katok and Lindenstrauss [10] is the *product structure* of leafwise measures discussed in [9].

Theorem 5.5 (Product structure [9, Thm. 8.4]). *Let $\{\alpha_i : i = 1, \dots, \ell\} \subseteq \Phi$ be pairwise inequivalent Lyapunov weights for a such that $\Psi_{a_0} = \{[\alpha_i] : i = 1, \dots, \ell\}$. Let $U \le G_{a_0}^-$ be a Zariski connected A-normalized subgroup. For every $i = 1, \dots, \ell$ define $U_i = U \cap G^{[\alpha_i]}$, and let*

$\phi : \prod_{i=1}^{\ell} U_i \to U$ be given by $\phi(u_1, \ldots, u_\ell) = u_1 \cdots u_\ell$. Then ϕ is a homeomorphism. Moreover, one has

$$\mu_x^U \propto \phi_*\big(\mu_x^{U\cap G^{[\alpha_1]}} \times \cdots \times \mu_x^{U\cap G^{[\alpha_\ell]}}\big) \quad \text{for } \mu\text{-a.e. } x \in X\ .$$

The product structure of the leafwise measures is relevant to our purposes in that it provides the following corollary, of which we sketch a proof.

Corollary 5.6. *Let μ be an a-invariant probability measure on $X = \Gamma\backslash G$. Let $\alpha_1, \cdots, \alpha_l \in \Psi_{a_0}$ be pairwise inequivalent Lyapunov weights such that $\Psi_{a_0} = \{[\alpha_i] : i = 1, \ldots, l\}$. Let $U \le G^-_{a_0}$ be a Zariski connected, a-normalized subgroup. Then the entropy contribution of U is given by*

$$h_\mu(a_0, U) = \sum_{i=1}^{l} h_\mu(a_0, U \cap G^{[\alpha_i]})\ .$$

Proof. Let

$$\mu = \int_X \mu_x^{\mathcal{E}} d\mu(x)$$

be an ergodic decomposition of μ with respect to the map T_{a_0}. Then the entropy contribution of U satisfies

$$h_\mu(a_0, U) = \int_X h_{\mu_x^{\mathcal{E}}}(a_0, U) d\mu(x),$$

where for any ergodic probability measure ν invariant under T_{a_0} (in particular, for any $\mu_x^{\mathcal{E}}$) we have

$$h_\nu(a_0, U) = \lim_{n\to\infty} -\frac{\log \nu_y^U(\theta_{a_0}^n(V))}{n}$$

for ν-almost every $y \in X$, where $V \subset U$ is an arbitrary bounded neighborhood of the identity. The previous equality holds since the function on the right-hand side is essentially T_{a_0}-invariant, and thus constant ν-almost everywhere by ergodicity (equal to its average over the whole space). As a consequence of the first assertion in Theorem 5.5, we may choose V of the form $V = V_1 \cdots V_l$, where V_i is a neighborhood of the identity in the subgroup $U \cap G^{[\alpha_i]}$ for each $i = 1, \ldots, l$. The product structure of leafwise measures (cf. Theorem 5.5) readily delivers

$$h_\nu(a_0, U, y) = \sum_{i=1}^{l} h_\nu(a_0, U \cap G^{[\alpha_i]}, y)$$

for μ-almost every $y \in X$, from which the corollary follows. □

6. The high entropy method and an inequality for entropy contribution

We preserve the same setup and notation of the previous section: $G < \mathbf{G}(\mathbb{Q}_S)$ is an S-algebraic group, $\Gamma < G$ an S-arithmetic subgroup, $X = \Gamma\backslash G$, $A < G$ a class-$\mathcal{A}'$ subgroup, $a_0 \in A \setminus \{e_G\}$, $G^-_{a_0}$ the stable horospherical subgroup defined by a_0, Φ_{a_0} the set of Lyapunov weights α for A satisfying $\alpha(a_0) < 0$.

6.1. **The high-entropy method.** The following result is one of the essential tools needed in the Section 7 in order to prove unipotent invariance.

Theorem 6.1 ([9, Thm. 8.5])**.** *Let $d \ge 2$ be an integer, $a\colon \mathbb{Z}^d \to G$ be a class-$\mathcal{A}'$ homomorphism, $A = a(\mathbb{Z}^d)$, μ an A-invariant and ergodic probability measure on $X = \Gamma\backslash G$, $a_0 \in A \setminus \{e_G\}$. Then, there exists Zariski connected, A-normalized subgroups $H < P$ of $G^-_{a_0}$ such that the following hold.*

(1) the support of $\mu_x^{G^-_{a_0}}$ is contained in P for μ-almost every $x \in X$.

(2) $\mu_x^{G^-_{a_0}}$ is bi-invariant under H.

(3) *H is normal in P, and if $\alpha, \beta \in \Phi_{a_0}$ are coarsely inequivalent and $g_\alpha \in P \cap G^{[\alpha]}$, $g_\beta \in P \cap G^{[\beta]}$, then $g_\alpha H$ and $g_\beta H$ commute in P/H.*
(4) *$\mu_x^{G^{[\alpha]}}$ is bi-invariant under $H \cap G^{[\alpha]}$ for all $\alpha \in \Phi_{a_0}$.*

Remark 6.2. Note that, as stated in [9], the subgroups H and P a priori depend on the basepoint x. Using the class-$\mathcal{A}'$ assumption and ergodicity of A acting on (X, μ), one finds by [73, Prop. 2.1.11] that the Lie algebras are μ-almost surely constant and therefore Zariski-connectedness of H and P implies the result as stated above.

6.2. **Entropy rigidity.** The second method by which we obtain additional invariance is as follows. We fix a compact metric space Ω with a $\mathbb{Z}^d$-action and consider the diagonal action $\tilde{a}$ of $\mathbb{Z}^d$ on $Z \times \Omega$. We recall the following result stated in [70, Rem. 2.5].

Proposition 6.3. *There exist a subgroup $\Sigma < \mathbb{Z}^d$ of finite index and a sequence of pairwise distinct, Σ-normalized subgroups*

$$\{\mathrm{id}\} = R_0 \lhd \cdots \lhd R_s = R_{\mathrm{u}}(\mathbb{Q}_S)$$

such that the following are true.

(1) *For all $1 \le i \le s$ the group R_i/R_{i-1} is abelian.*
(2) *For all $0 \le i \le s$ the group $\Lambda \cap R_i < R_i$ is a lattice.*
(3) *For all $1 \le i \le s$ the action of Σ on $F_i = R_i/(\Lambda \cap R_i)R_{i-1}$ is irreducible. That is, for every finite index subgroup $\Sigma' < \Sigma$ there are no proper infinite closed invariant subgroups of F_i.*

For what follows, we fix a sequence $\{R_i\}_{i=1}^s$ and a subgroup $\Sigma < \mathbb{Z}^d$ as in Proposition 6.3. We denote by $a_{i,\mathrm{ab}}$ the induced action of Σ on F_i and we let λ_i (or λ, when the index is clear from context) denote the Haar measure on F_i.

We let R_i act on $Z_{i-1} = Z/R_{i-1}$ and we note that the orbit of any G-translate of the identity coset identifies with F_i. We equip Z_i with the push forward μ_i of μ under the quotient map. This way we obtain a sequence of factors $Z_{i-1} \to Z_i$. By the Abramov-Rokhlin summation formula, for all $\mathbf{n} \in \Sigma$ we have

$$h_\mu\big(a(\mathbf{n})\big) = \sum_{i=0}^{s-1} h_\mu\big(a_i(\mathbf{n})|Z_{i+1}\big),$$

where a_i denotes the induced action of Σ on Z_i.

For the remainder of this section, we fix an element $\mathbf{n}_0 \in \Sigma \setminus \{0\}$. As m_Y is a measure of maximal entropy, we know from the Abramov-Rokhlin formula (7.1) that there exists a minimal $0 \le i^* < s$ such that

$$h_\mu\big(a_{i^*}(\mathbf{n}_0)|Y_{i^*+1}\big) > 0.$$

In what follows, we denote by $\tilde{a}$ the diagonal action of Σ on $Z \times \Omega$ and we assume that $\tilde{\mu}$ is a Σ^d-invariant measure on $Z \times \Omega$.

Theorem 6.4. *We have $h_\lambda\big(a_{i^*,\mathrm{ab}}(\mathbf{n}_0)\big) > 0$. Moreover, let $U < H^-_{\mathbf{n}_0}$ be a Zariski-closed $a(\mathbf{n}_0)$-normalized subgroup, then*

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0), U|Z_{i^*+1} \times \Omega\big) \le \frac{h_\lambda\big(a_{i^*,\mathrm{ab}}(\mathbf{n}_0), U \cap R_{\mathrm{i}^*+1}\big)}{h_\lambda\big(a_{i^*,\mathrm{ab}}(\mathbf{n}_0)\big)} h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0)|Z_{i^*+1} \times \Omega\big).$$

Proof. We show that

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0), U|Z_{i^*+1} \times \Omega\big) = h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0), U \cap R_{i^*+1}|Z_{i^*+1} \times \Omega\big).$$

To this end we recall that

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0), U|Z_{i^*+1} \times \Omega\big) = H_{\tilde{\mu}}\big(\tilde{\mathcal{C}}_U|\tilde{a}(\mathbf{n}_0)^{-1}\tilde{\mathcal{C}}_U \vee \tilde{\mathcal{B}}_{i^*+1}\big),$$

where $\tilde{\mathcal{C}}_U$ is a countably generated, $\tilde{a}(\mathbf{n}_0)$-decreasing, U-subordinate σ-algebra on $Z \times \Omega$, and $\tilde{\mathcal{B}}_{i^*+1} = \mathcal{B}_{i^*+1} \otimes \mathcal{B}_\Omega$ for $\mathcal{B}_\Omega$ the Borel σ-algebra on Ω and $\mathcal{B}_{i^*+1}$ the preimage of the Borel σ-algebra on Z_{i^*+1} under the canonical projection $Z \to Z_{i^*+1}$. Note that the atoms of $\mathcal{B}_\Omega$ are singletons and the atoms of $\mathcal{B}_{i^*+1}$ are R_{i^*+1}-orbits. As U acts trivially on Ω, we have that $\tilde{\mathcal{C}}_U = \mathcal{C}_U \otimes \mathcal{B}_\Omega$, where $\mathcal{C}_U$ is a countably generated, $a(\mathbf{n}_0)$-decreasing, U-subordinate σ-algebra on Z.The σ-algebra $\tilde{\mathcal{B}}_{i^*+1}$ is $\tilde{a}(\mathbf{n}_0)$-invariant. In particular, we have

$$\begin{aligned} h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0), U|Z_{i^*+1} \times \Omega\big) &= H_{\tilde{\mu}}\big(\tilde{\mathcal{C}}_U \vee \tilde{\mathcal{B}}_{i^*+1} | a(\mathbf{n}_0)^{-1}\tilde{\mathcal{C}}_U \vee \tilde{\mathcal{B}}_{i^*+1}\big) \\ &= H_{\tilde{\mu}}\big(\tilde{\mathcal{C}}_U \vee \tilde{\mathcal{B}}_{i^*+1} | a(\mathbf{n}_0)^{-1}(\tilde{\mathcal{C}}_U \vee \tilde{\mathcal{B}}_{i^*+1})\big) \end{aligned}$$

Note that for all $z \in Z$ we have

$$[z]_{\mathcal{C}_U \vee \mathcal{B}_{i^*+1}} = [z]_{\mathcal{C}_U} \vee [z]_{\mathcal{B}_{i^*+1}} = [z]_{\mathcal{C}_U} \cap z R_{i^*+1}$$

and therefore $\mathcal{C}_U \vee \mathcal{B}_{i^*+1}$ is $U \cap R_{i^*+1}$-subordinate. Now we proceed like in [18]. □

Lemma 6.5 (cf. [18, Lem. 7.2])**.** *In the notation introduced above, assume that $\tilde{\mu}$ projects to μ. Let $[\chi]$ be a coarse Lyapunov exponent for F_{i^*+1}. Then there is $\kappa_{\tilde{\mu},\Omega,[\chi]} \geq 0$ such that for all $\mathbf{n} \in \Sigma$*

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}), H^{[\chi]}|Z_{i^*+1} \times \Omega\big) = \kappa_{\tilde{\mu},\Omega,[\chi]} h_\lambda\big(a_{i^*,\mathrm{ab}}(\mathbf{n}), H^{[\chi]}\big).$$

Proof. The argument is verbatim as in [18]. Positivity of c_λ follows by the maximal entropy assumption. □

Lemma 6.6 (cf. [18, Lem. 7.3])**.** *In the notation introduced above, assume that $\tilde{\mu}$ projects to μ. Let $[\chi]$ be a coarse Lyapunov weight for Z. Then*

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0), H^{[\chi]} \cap R_{i^*}|Z_{i^*+1} \times \Omega\big) = 0.$$

Moreover, if $[\chi]$ is not a coarse Lyapunov weight for F_{i^*+1}, then

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}_0)|Z_{i^*+1} \times \Omega\big) = 0.$$

Combining all the above with the product structure, we find the following theorem.

Theorem 6.7. *In the notation above, assume that $\tilde{\mu}$ projects to μ. There exists $\kappa_{\tilde{\mu},\Omega} > 0$ such that*

$$h_{\tilde{\mu}}\big(\tilde{a}(\mathbf{n}), H^{[\chi]}|Z_{i^*+1} \times \Omega\big) = \kappa_{\tilde{\mu},\Omega} h_\lambda\big(a_{i^*+1,\mathrm{ab}}(\mathbf{n}), H^{[\chi]} \cap R_{\mathrm{u}}(\mathbb{Q}_S)\big).$$

7. Producing additional unipotent invariance

This section is devoted to showing that a $\mathbb{Z}^d$-invariant and ergodic joining μ of m_X and m_Y, as in the statement of Theorem 1.3, has in fact to exhibit some additional unipotent invariance. This will crucially enable us to resort to Proposition 1.4 in order to show triviality of μ. The latter part of the argument will be carried out in detail in the next section.

Notation and setup. We briefly recall the setup for our main result (cf. Theorem 1.1), which will be fixed until the end of the paper. Let $\mathbf{B}, \mathbf{G}$ be respectively solvable and perfect connected linear algebraic groups defined over $\mathbb{Q}$, S a finite set of places of $\mathbb{Q}$ containing the infinite place. Let $G < \mathbf{G}(\mathbb{Q}_S)$ and $B < \mathbf{B}(\mathbb{Q}_S)$ S-algebraic groups Assume $\Gamma < G$ and $\Lambda < B$ are S-arithmetic subgroups, and denote by $X = \Gamma\backslash G$ and $Y = \Lambda\backslash B$ the corresponding S-arithmetic quotients.

We further let $a_G \colon \mathbb{Z}^d \to G$, $a_B \colon \mathbb{Z}^d \to \Lambda$ be two class-$\mathcal{A}'$ homomorphisms, where $d \geq 2$ is an integer. Then $\mathbb{Z}^d$ acts measure-preservingly on X with respect to the Haar measure m_X via $\mathbf{n} \cdot \Gamma g = \Gamma g a_G(-\mathbf{n})$, for $\mathbf{n} \in \mathbb{Z}^d$ and $g \in G$; similarly there is an induced $\mathbb{Z}^d$-action on Y, and we fix a $\mathbb{Z}^d$-invariant probability measure m_Y of maximal entropy with respect to the action of $a_B(\mathbb{Z}^d)$. Denote by $a =: \mathbb{Z}^d \to G \times B$ the diagonal homomorphism defined by $a(\mathbf{n}) = \big(a_G(\mathbf{n}), a_B(\mathbf{n})\big)$ for all $\mathbf{n} \in \mathbb{Z}^d$, and finally let μ be a $\mathbb{Z}^d$-invariant and ergodic joining of m_X and m_Y on the product space $X \times Y$, which we identify canonically with the quotient $\Delta\backslash H$, where $H = G \times B$ and $\Delta = \Gamma \times \Lambda$.

As already mentioned, we set out to prove the following theorem.

Theorem 7.1 (Unipotent invariance)**.** *Suppose the assumptions in Theorem 1.1 hold, and let notation be as above. Then, there exists a non-trivial Zariski connected unipotent subgroup $U < H$ generated by one-parameter unipotent subgroups and normalized by $a(\mathbb{Z}^d) < H$ such that μ is invariant under U.*

In what follows, we denote by Ψ_X and Ψ_Y the set of *non-trivial* coarse Lyapunov weights associated to a_G and a_B respectively. We have to distinguish between two cases, namely either $\Psi_X = \Psi_Y$ or $\Psi_X \neq \Psi_Y$.

7.1. **Case 1: the Lyapunov weights do not match.** This case is considerably easier to deal with and more general: unipotent invariance is established through a version of the Abramov-Rokhlin formula for conditional entropy; cf. [51, Sec. 6.1]. Also, we crucially use that for any $[\alpha] \in \Psi_X \cup \Psi_Y$ we have

$$H^{[\alpha]} = G^{[\alpha]} \times B^{[\alpha]},$$

the verification of which is purely formal and hence left to the reader. Note that the argument to follow does not rely on any higher rank assumption on a_G or a_B. Specifically, we obtain the following corollary to [17, Prop. 6.5]:

Corollary 7.2. *Let $\alpha\colon \mathbb{Z}^d \to \mathbb{R}$ be such that $[\alpha] \in (\Psi_X \setminus \Psi_Y) \cup (\Psi_Y \setminus \Psi_X)$. Then the joining μ is invariant under the subgroup $H^{[\alpha]}$, which is a non-trivial, Zariski closed, connected unipotent subgroup of H.*

Proof. For simplicity, assume $[\alpha] \in \Psi_Y \setminus \Psi_X$; the other case is exactly analogous.

Let $N = G \times \{e\} \subset H$ be the subgroup defined as the image of the canonical embedding of G in H, and denote by $\pi\colon H \to H/N \simeq B$ the canonical projection. Clearly, $\Delta \cap N \simeq \Gamma$ is a lattice in N. It follows from [17, Prop. 6.5] that, for each $\mathbf{n} \in \mathbb{Z}^d$,

$$h_\mu\big(a(\mathbf{n}), H^{[\alpha]}\big) = h_{m_Y}\big(a_B(\mathbf{n}), B^{[\alpha]}\big) + h_\mu\big(a(\mathbf{n}), H^{[\alpha]} \cap N\big)\ . \tag{7.1}$$

Since $[\alpha] \notin \Psi_X$, we have $H^{[\alpha]} = \{e\} \times B^{[\alpha]}$ and the group $H^{[\alpha]} \cap N$ is trivial. Therefore, the second summand in (7.1) vanishes and

$$h_\mu\big(a(\mathbf{n}), H^{[\alpha]}\big) = h_{m_Y}\big(a_B(\mathbf{n}), B^{[\alpha]}\big).$$

Furthermore, we note that

$$h_\mu\big(a(\mathbf{n}), H^{[\alpha]}\big) = h_{m_Y}\big(a_B(\mathbf{n}), B^{[\alpha]}\big), \tag{7.2}$$

as any $H^{[\alpha]}$-subordinate σ-algebra on $X \times Y$ is of the form $\mathcal{B}_X \otimes \mathcal{A}$, where $\mathcal{B}_X$ is the Borel σ-algebra on X and $\mathcal{A}$ is a $B^{[\alpha]}$-subordinate σ-algebra on Y.

As m_Y is a measure of maximal entropy, it follows that $h_\mu\big(a(\mathbf{n}), H^{[\alpha]}\big)$ is maximal; by virtue of [16, Thm. 7.9], this implies that μ is $H^{[\alpha]}$-invariant. It was already shown in Section 3.1 that $H^{[\alpha]}$ is unipotent, Zariski closed, and connected. □

7.2. **Case 2: the Lyapunov weights match perfectly.** The second case, in which the Lyapunov weights for a_G and a_B coincide, is more involved; the proof of invariance of μ under a unipotent subgroup relies here on the *high entropy method* and on *rigidity of the entropy function* outlined in Section 6. Our treatment follows [17, Sec. 7] and [18, Sec. 7] verbatim, up to minor modifications required to fit our setup.

Denote by Φ_H the set of Lyapunov weights for the homomorphism $a\colon \mathbb{Z}^d \to H$, and by Ψ_H the corresponding set of coarse Lyapunov weights. As explained in [17, Sec. 7.2], we may assume that there exists $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$ and some $\alpha_0 \in \Phi_H$ such that $\alpha_0(\mathbf{n}) = 0$. We also define

$$P^- = \prod_{[\alpha]\in\Psi_H,\, \alpha(\mathbf{n})<0} P^{[\alpha]} \quad \text{and} \quad P^+ = \prod_{[\alpha]\in\Psi_H,\, \alpha(\mathbf{n})>0} P^{[\alpha]},$$

where $P^{[\alpha]} < H^{[\alpha]}$ is the minimal Zariski-closed, connected subgroup of $H^{[\alpha]}$ such that $\operatorname{supp}\mu_z^{H^\alpha} \subset P^{[\alpha]}$ for μ-almost every $z \in \Delta\backslash H$. Then P^- is the minimal Zariski-closed subgroup of

$$H^- = \{h \in H : a(\mathbf{n})^k h a(-\mathbf{n})^k \to e_H \text{ as } k \to \infty\}$$

containing $\operatorname{supp}\mu_z^{H^-}$ for μ-almost every $z \in \Delta\backslash H$. An analogous statement holds for P^+ (with respect to H^+ defined changing the roles of $a(-\mathbf{n})$ instead of $a(\mathbf{n})$).

The discussion now splits into two cases, depending on whether $P^{[\alpha_0]}$ centralizes both P^- and P^+, in which case we invoke *rigidity of the entropy function* (cf. [18, Sec. 7]; otherwise, we appeal to the *high entropy method* (cf. [8–10]).

Invariance via the high entropy method. We first discuss the case in which $P^{[\alpha_0]}$ does not centralize either P^- or P^+. The argument being the same for both cases, we assume without loss of generality that $P^{[\alpha_0]}$ does not commute with P^-. Using the higher rank assumption, we can choose $\mathbf{n}_0 \in \mathbb{Z}^d$ such that $\alpha_0(\mathbf{n}_0) < 0$ and

$$\alpha(\mathbf{n}) < 0 \Longrightarrow \alpha(\mathbf{n}_0) < 0 \text{ for all } \alpha \in \Phi_H,$$

that is, $\alpha(\mathbf{n}_0) < 0$ for all α appearing in the definition of P^-. Letting $\mathbf{a} = a(\mathbf{n}_0)$, we have that $H^{[\alpha_0]} \cup P^- \subset H^-_{\mathbf{a}}$. We let $H_0 \lhd P_0 < H^-_{\mathbf{a}}$ be the Zariski connected, a-normalized subgroups from Theorem 6.1; hence, for μ-almost all $z \in \Delta\backslash H$, we have that $\operatorname{supp}\mu_z^{H^-_{\mathbf{a}}} \subset P_0$, $\mu_z^{H^-_{\mathbf{a}}}$ is H_0-invariant, and $\mu_z^{H^{[\alpha]}}$ is $H_0 \cap H^{[\alpha]}$-invariant. Moreover, we know that for inequivalent weights $\alpha, \beta \in \Phi_{\mathbf{a}}$ and $h_\alpha \in P_0 \cap H^{[\alpha]}$, $h_\beta \in P_0 \cap H^{[\beta]}$, we have $[h_\alpha, h_\beta] \in H_0$. We know from Theorem 5.5 that

$$\mu_z^{H^-_{\mathbf{a}}} \propto \prod_{[\alpha]\in\Psi_H:\alpha(\mathbf{n}_0)<0} \mu_z^{H^-_{\mathbf{a}}\cap H^{[\alpha]}} \propto \prod_{[\alpha]\in\Psi_H:\alpha(n_0)<0} \mu_z^{H^-_{\mathbf{a}}\cap P^{[\alpha]}}$$

and thus from Theorem 6.1 that for μ-almost every $z \in \Delta\backslash H$ the measure $\mu_z^{H^-_{\mathbf{a}}}$ is bi-invariant under the group generated by all commutators $[h_1, h_2] \in H_0$ with $h_1 \in H^{[\alpha_0]}$ and $h_2 \in H^{[\alpha]}$ for $\alpha \in \Phi_H \setminus [\alpha_0]$ such that $\alpha(\mathbf{n}_0) < 0$. Notice that $[h_1, h_2] \in H^-_{\mathbf{a}}$; as $P^{[\alpha_0]}$ does not commute with P^-, we have found a non-trivial Zariski-closed unipotent subgroup $U < H_0 < H$ such that μ is U-invariant.

Invariance via rigidity of the entropy function. It remains to deal with the case where $P^{[\alpha_0]}$ commutes with both P^- and P^+. We fix a coarse Lyapunov weight $[\alpha]$ occuring in the definition of P^-, say, and we define Ω to be the space of equivalence classes of locally finite Radon-measures on $U = P^{[\chi]} \cap R_{\mathrm{u}}(\mathbb{Q}_S)$ which are integrable for some suitably chosen non-negative measureable function f. The space Ω then becomes a compact metric space and for the right choice of f we find that $x \mapsto [\mu_x^U]$ has image in Ω for μ-almost everywhere. We let $\mathbb{Z}^d$ act trivially on Ω.

Proposition 7.3. *In the notation of Theorem 6.7, we have $\kappa_{\tilde{\mu},\Omega} = \kappa_\mu$, where $\kappa_\mu := \kappa_{\tilde{\mu},\{\cdot\}}$.*

Proof. Here we use that $P^{[\alpha_0]}$ commutes with P^- to adapt the proof of [18, Prop. 7.5] verbatim. □

In combination with the product structure, we obtain the following corollary.

Corollary 7.4. *Fix $\mathbf{n} \in \mathbb{Z}^d$ and a coarse Lyapunov weight χ satisfying $\chi(\mathbf{n}) < 0$. Let $W = H^{[\chi]} \cap R_{\mathrm{u}}(\mathbb{Q}_S)$. For any subset $Z' \subseteq Z$ of full measure, there exist $z \in Z'$ and $w \in W \setminus \{\mathrm{id}\}$ such that $zw \in Z'$ and $\mu_z^W \propto \mu_{zw}^W$.*

From here one can deduce additional invariance as outlined in [18, §7].

Conclusion. To sum up the result of this section, we have obtained that our joining μ is invariant under the subgroup $\langle U, a(\mathbb{Z}^d)\rangle \simeq U \rtimes a(\mathbb{Z}^d)$ generated by the unipotent subgroup U and the diagonalizable subgroup $a(\mathbb{Z}^d)$. Observe in addition that, since we are assuming from the outset that μ is ergodic under the action of $a(\mathbb{Z}^d)$, it is *a fortiori* ergodic for the action of the bigger group $\langle U, a(\mathbb{Z}^d)\rangle$. This brings us in a position to apply the measure classification result in Proposition 1.4, which alongside the joining structure of μ will lead us to the conclusion of the proof of Theorem 1.1.

8. Disjointness: proof of the main theorem

The final section of this article is consecrated to the proof of Theorem 1.3, from which the more general Theorem 1.1 follows as explicated in Section 4.2.

Let thus μ be a $\mathbb{Z}^d$-invariant and ergodic joining of m_X and m_Y on $X \times Y \simeq \Delta\backslash H$. We may assume additional unipotent invariance of μ, which has been established in Theorem 7.1; more precisely, μ is invariant under a non-trivial subgroup $U < H$ generated by one-parameter unipotent subgroups and normalized by the class-$\mathcal{A}'$ group $A = a(\mathbb{Z}^d)$. In view of Proposition 1.4, there exists a Zariski-connected algebraic $\mathbb{Q}$-subgroup $\mathbf{L} < \mathbf{G}$ of class $\mathcal{F}$, a finite-index subgroup $L < \mathbf{L}(\mathbb{Q}_S)$ and an element $h_0 = (g_0, b_0) \in H$, with $\Delta h_0 \in \operatorname{supp}\mu$, such that $U < h_0^{-1} L h_0$, μ is invariant under $h_0^{-1} L h_0$ and is concentrated on the orbit $\Gamma N_H^1(\mathbf{L}(\mathbb{Q}_S))h_0$, where we recall that $N_H^1(\mathbf{L}(\mathbb{Q}_S))$ is the set of elements in H normalizing $\mathbf{L}(\mathbb{Q}_S)$ whose action by conjugation preserves the Haar measure on it. Let μ' be the push-foward under of μ under the action of h_0. Then μ' is an A'-invariant and ergodic joining of m_X and m_Y, where $A' = h_0 A h_0^{-1}$; furthermore, μ' is invariant under L and concentrated on the $N_H^1(\mathbf{L}(\mathbb{Q}_S))$-orbit of the identity coset $\Delta \in \Delta\backslash H$. Postcomposing the homomorphism a with conjugation by h_0 doesn't alter the properties of a_G and a_B; therefore, just as before, we may assume without loss of generality that h_0 is the identity element of H, upon replacing μ by μ'.

We now make use of the joining assumption to pin down the algebraic structure of the normalizer $N_{\mathbf{H}}(\mathbf{L})$ of $\mathbf{L}$ inside $\mathbf{H}$. Recall that the projection of any Zariski-connected $\mathbb{Q}$-subgroup of $\mathbf{H}$ onto each of the factors $\mathbf{G}$ and $\mathbf{B}$ is a Zariski-connected $\mathbb{Q}$-group ([65, Prop. 2.2.5]).

Proposition 8.1. *Let $\pi_{\mathbf{G}} \colon \mathbf{H} \to \mathbf{G}$ deote the canonical projection map. Then $\pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L})) = \mathbf{G}$.*

Proof. Since $\mu(\Delta N_H^1(\mathbf{L}(\mathbb{Q}_S))) = 1$ and $N_H^1(\mathbf{L}(\mathbb{Q}_S)) < N_{\mathbf{H}(\mathbb{Q}_S)}(\mathbf{L}(\mathbb{Q}_S)) = (N_{\mathbf{H}}(\mathbf{L}))(\mathbb{Q}_S)$, it follows by projecting onto X that the orbit of the identity coset $\Gamma \in X$ under the closed projection $G \cap \pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L}))(\mathbb{Q}_S)$ of the group $H \cap N_{\mathbf{H}}(\mathbf{L})(\mathbb{Q}_S)$ to G has full m_X-measure. As shown in [64, Lem. 2.2], this implies that such orbit is the full space X. Lattices in second countable topological groups being at most countable, it follows that $G \cap \pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L}))(\mathbb{Q}_S)$ has at most countable index in G, and the same holds true for $\pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L}))(\mathbb{Q}_S)$ in $\mathbf{G}(\mathbb{Q}_S)$, since G has finite index in $\mathbf{G}(\mathbb{Q}_S)$. This delivers $\pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L})) = \mathbf{G}$, as desired; otherwise, since $\mathbf{G}$ is Zariski-connected, a proper inclusion $\pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L})) \lneq \mathbf{G}$ would force the quotient $\mathbb{Q}$-variety $\mathbf{G}/\pi_{\mathbf{G}}(N_{\mathbf{H}}(\mathbf{L}))$ to have strictly positive dimension (cf. [65, Cor. 5.5.6]), contradicting the fact that its set of $\mathbb{Q}_S$-points is at most countable. ☐

At this point, we need to recall Goursat's lemma ([26]) from abstract group theory:

Proposition 8.2 (Goursat's lemma)**.** *Let G_1, G_2 be two groups, H a subgroup of $G_1 \times G_2$ projecting surjectively onto both factors. Define*

$$N_1 = \{g_1 \in G_1 : (g_1, e_{G_2}) \in H\},\ N_2 = \{g_2 \in G_2 : (e_{G_1}, g_2) \in H\},$$

where e_{G_1}, e_{G_2} denote the identity elements of G_1, G_2, respectively. Then, N_i is a normal subgroup of G_i, $i = 1, 2$, and the image of H under the canonical map $G_1 \times G_2 \to G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $\phi\colon G_1/N_1 \to G_2/N_2$.

A consequence of Goursat's lemma is that, if G_1 and G_2 have no non-trivial isomorphic factors, then the only subgroup of $G_1 \times G_2$ projecting surjectively onto both factors is the full direct product $G_1 \times G_2$. Since the classes of perfect and solvable groups are both closed under taking quotients, and a non-trivial perfect group cannot be isomorphic to a solvable group, we conclude:

Corollary 8.3. *Let $\mathbf{B}'$ be the projection of $N_{\mathbf{H}}(\mathbf{L})$ to $\mathbf{B}$. Then $N_{\mathbf{H}}(\mathbf{L}) = \mathbf{G} \times \mathbf{B}'$.*

Replacing $\mathbf{B}$ by $\mathbf{B}'$ if needed[15], we may now assume that $\mathbf{L}$ is a normal $\mathbb{Q}$-subgroup of $\mathbf{H}$.

The upshot of the foregoing discussion is that the ergodic joining μ is invariant (up to finite-index issues) under the group of $\mathbb{Q}_S$-points of a non-trivial, normal $\mathbb{Q}$-subgroup of $\mathbf{H} = \mathbf{G} \times \mathbf{B}$. In case $\mathbf{G}$ is semisimple, Proposition 8.4 in the upcoming section, in conjuction with the classification of normal algebraic subgroups of semisimple groups given in Corollary 2.3, allows for a neat description of the whole range of possibilities for the group $\mathbf{L}$. An inductive argument on the number of simple factors of $\mathbf{G}$ and on the dimension of $R_{\mathrm{u}}(\mathbf{B})$ as an algebraic variety leads to the conclusion of the proof, as explained in Section 8.1. In order to deal with the general case of $\mathbf{G}$ perfect, we rely on an argument inspired by the statement, and by the proof, of [17, Thm. 1.6]; this is carried out in Section 8.2, thereby achieving the proof of Theorem 1.3.

8.1. **The case G semisimple.** Throughout this subsection, $\mathbf{G}$ is assumed to be semisimple. In this case, there is a simple description of all normal algebraic subgroups of the product $\mathbf{H} = \mathbf{G} \times \mathbf{B}$: they all split as products of normal subgroups of the two factors:

Proposition 8.4. *Let $\mathbf{G}, \mathbf{B}$ be, respectively, a semisimple and a solvable Zariski-connected linear algebraic group defined over $\mathbb{Q}$, and let $\mathbf{L} < \mathbf{G} \times \mathbf{B}$ be a Zariski-connected normal $\mathbb{Q}$-subgroup of class $\mathcal{F}$. Then $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$, where $\mathbf{L}_1 < \mathbf{G}$ and $\mathbf{L}_2 < R_u(\mathbf{B})$ are Zariski-connected normal $\mathbb{Q}$-subgroups of $\mathbf{G}$ and $\mathbf{B}$, respectively.*

The proof of this proposition is straightforward; details are relegated to Appendix B.

The semisimple $\mathbb{Q}$-group $\mathbf{G}$ is an almost direct product of its $\mathbb{Q}$-almost simple factors $\mathbf{G}_1, \dots, \mathbf{G}_r$ (cf. Theorem 2.2); we assume first that $r = 1$, that is $\mathbf{G}$ is $\mathbb{Q}$-almost simple.

We distinguish two cases: either $\mathbf{L}_2$ is trivial or it is not. Suppose first that $\mathbf{L}_2$ is non-trivial. The group $\mathbf{L}_2(\mathbb{Q}_S) < L$, which is contained in L since by Lemma 2.5 unipotent groups do not admit any proper finite-index subgroup, acts with compact orbits on $X \times Y$[16]; let q denote the canonical projection $H \to H/(\mathbf{L}_2(\mathbb{Q}_S))$.

The proof of the following lemma is immediate from Propositions 5.3 and 5.4.

Lemma 8.5. *The projection $q_*\mu$ of μ to the quotient homogeneous space $\Delta\backslash H/\mathbf{L}_2(\mathbb{Q}_S)$ has maximal entropy for the homomorphism $\mathbb{Z}^d \xrightarrow{a} H \xrightarrow{\overline{\pi}} H/\mathbf{L}_2(\mathbb{Q}_S)$, where $\overline{\pi}$ is the canonical projection map.*

By virtue of Proposition A.7, we might harmlessly replace μ by its projection onto the double quotient $\Delta\backslash H/\mathbf{L}_2(\mathbb{Q}_S)$; on account of Lemma 8.5, this projection is a measure satisfying all our current assumptions, with $\mathbf{B}$ replaced by $\mathbf{B}/\mathbf{L}_2$. If Theorem 1.3 is shown to hold for such a projection, then it holds for μ as well, as the conclusion of Proposition A.7 shows.

[15] Observe that Propositon 1.4 furnishes, in particular, that A is contained in $N_H^1(\mathbf{L}(\mathbb{Q}_S))$, hence in the larger group $H \cap (\mathbf{G} \times \mathbf{B}')(\mathbb{Q}_S)$.

[16] The orbit of the identity coset is compact since $\mathbf{L}_2$ is unipotent and hence clearly anisotropic over $\mathbb{Q}$ (cf. [40, Thm. 3.2.4(b)]); the same property transfers immediately to all other orbits, which are obtained by translation of the identity orbit, for $\mathbf{L}_2(\mathbb{Q}_S)$ is normal in $H = G \times B$.

Therefore, we are left to deal with the case $\mathbf{L} = \mathbf{L}_1$. The subgroup $\mathbf{L} < \mathbf{G}$ is a non-trivial Zariski-connected normal $\mathbb{Q}$-subgroup of the $\mathbb{Q}$-almost simple group $\mathbf{G}$, so that necessarily $\mathbf{L} = \mathbf{G}$. The projection L_1 of the finite-index subgroup $L < \mathbf{L}(\mathbb{Q}_S)$ onto G is a finite-index subgroup of G; we identify it with its isomorphic copy inside $H = G \times B$, and recall that μ is, in particular, invariant under L_1.

Remark 8.6. At this point, knowing in addition that $L_1 = G$ would readily yield the conclusion $\mu = m_X \times m_Y$. This is a consequence of the following general observation: if R is a (abstract) group acting by measurable transformations of measurable space X, Y is a second measurable space, μ is a probability measure on $X \times Y$ invariant under the R-action $r \cdot (x, y) = (r \cdot x, y)$, and the projection ρ of μ to X is the unique R-invariant probability measure on X, then[17] $\mu = \rho \times \nu$, where ν is the projection of μ to Y.

The general case of a finite-index subgroup $L_1 < G$ requires an ergodic-decomposition argument, as outlined below; we shall make use, in particular, of a slight generalization of the previous observation, stated in Proposition 8.7.

Let $A_{L_1} = \{a = (a_1, a_2) \in A < G \times B : a_1 \in L_1\}$; as L_1 has finite index in G, A_{L_1} has finite index in A, which, together with the fact that A is commutative and normalizes L_1, implies that $A_{L_1}L_1$ is a normal subgroup of finite index of the locally compact[18] group AL_1. The quotient $AL_1/A_{L_1}L_1$ is a finite abelian group (isomorphic to a quotient of A/A_{L_1}). Proposition 4.6 gives thus that

$$\mu = \frac{1}{[AL_1 : A_{L_1}L_1]} \sum_{aA_{L_1}L_1 \in AL_1/A_{L_1}L_1} a_*\mu_0 \tag{8.1}$$

is an $A_{L_1}L_1$-ergodic decomposition of the AL_1-invariant and ergodic measure μ, where μ_0 is a given $A_{L_1}L_1$-ergodic component of μ. Projecting (8.1) to X, we obtain that

$$m_X = \frac{1}{[AL_1 : A_{L_1}L_1]} \sum_{(a_1,a_2)A_{L_1}L_1 \in AL_1/A_{L_1}L_1} (a_1)_*(\pi_X)_*\mu_0$$

is a $\pi_G(A_{L_1}L_1)$-ergodic decomposition of m_X, where $\pi_G \colon G \times B \to G$ is the canonical projection onto the first factor. On the one hand, observe that $\pi_G(A_{L_1}L_1) = L_1$, by definition of A_{L_1}; on the other hand, notice that $\pi_G(A_{L_1})$ acts ergodically with respect to m_X on X, being a finite-index subgroup of $a_G(\mathbb{Z}^d)$ (cf. Remark 1.2). *A fortiori*, the larger subgroup L_1 acts ergodically with respect to m_X. Uniqueness of the ergodic decomposition thus forces $m_X = (\pi_X)_*\mu_0$.

The following proposition allows to deduce that $\mu_0 = m_X \times (\pi_Y)_*\mu_0$. We formulate and prove it in the utmost generality:

Proposition 8.7. *Let X, Y be standard Borel spaces, R a locally compact group acting measurably on X, and trivially on Y. Let μ be a probability measure on $X \times Y$, which is invariant under the diagonal action of R on the product. Suppose R acts ergodically on X with respect to the R-invariant measure $\rho = (\pi_X)_*\mu$. Then $\mu = \rho \times \nu$, where ν is the projection of μ to Y.*

Proof. Upon choosing topological models (cf. Section 4.1) for the two actions, we may assume that X, Y are compact metrizable spaces on which R acts continuously. Let $(Z, \lambda) \ni z \mapsto \mu_z \in \mathcal{M}^1(X \times Y)$ be an R-ergodic decomposition of μ. Then $z \mapsto (\pi_X)_*\mu_z$ is an R-ergodic

[17] By the monotone class lemma, it suffices to show that $\mu(E \times F) = \rho(E)\nu(F)$ for any measurable sets $E \subset X, F \subset Y$. If $\nu(F) = 0$, then $\mu(E \times F) \leq \mu(X \times F) = \nu(F) = 0$, thus equality holds. If $\nu(F) > 0$; the probability measure ρ_F on X defined by $\rho_F(E) = \mu(E \times F)/\nu(F)$, for any measurable $E \subset X$, is R-invariant by R-invariance of μ and the fact that R acts trivially on Y. This forces $\rho_F = \rho$, that is $\mu(E \times F) = \rho(E)\nu(F)$ for any measurable $E \subset X$.

[18] Here the group AL_1 is meant to be endowed with the final topology for the product map $A \times L_1 \to AL_1$, where L_1 has the induced topology from G and A is equipped with the discrete topology.

decomposition of ρ and $z \mapsto (\pi_Y)_*\mu_z$ is an R-ergodic decomposition of ν. The assumptions on the measure-preserving R-actions on (X, ρ) and (Y, ν) imply that

$$\text{for } \lambda\text{-almost every } z \in Z, \quad (\pi_X)_*\mu_z = \rho \text{ and } (\pi_Y)_*\mu_z = \delta_{y(z)} \text{ for some } y(z) \in Y,$$

where the assignment $z \mapsto y(z)$ satisfies $\nu = \int_Z \delta_{y(z)} d\lambda(z)$. As a consequence, $\mu_z = \rho \times \delta_{y(z)}$ for λ-almost every $z \in Z$, and an application of Fubini's theorem gives

$$\mu = \int_Z \mu_z \, d\lambda(z) = \int_Z \rho \times \delta_{y(z)} d\lambda(z) = \rho \times \int_Z \delta_{y(z)} d\lambda(z) = \rho \times \nu,$$

as desired. □

We may now conclude, by virtue of Proposition 8.7, that

$$\begin{aligned}
\mu &= \frac{1}{[AL_1 : A_{L_1}L_1]} \sum_{(a_1,a_2)A_{L_1}L_1 \in AL_1/A_{L_1}L_1} (a_1)_* m_X \times (a_2)_*(\pi_Y)_*\mu_0 \\
&= \frac{1}{[AL_1 : A_{L_1}L_1]} \sum_{(a_1,a_2)A_{L_1}L_1 \in AL_1/A_{L_1}L_1} m_X \times (a_2)_*(\pi_Y)_*\mu_0 \\
&= m_X \times \left(\frac{1}{[AL_1 : A_{L_1}L_1]} \sum_{(a_1,a_2)A_{L_1}L_1 \in AL_1/A_{L_1}L_1} (a_2)_*(\pi_Y)_*\mu_0 \right) \\
&= m_X \times m_Y,
\end{aligned}$$

where the last equality follows by projecting (8.1) to Y.

To recap, we have proven Theorem 1.3 in the case $\mathbf{G}$ is $\mathbb{Q}$-almost simple. Now suppose inductively that the statement holds for any semisimple $\mathbb{Q}$-group with $s \leq r-1$ $\mathbb{Q}$-almost simple factors, and assume that $\mathbf{G}$ has r $\mathbb{Q}$-almost simple factors $\mathbf{G}_1, \ldots, \mathbf{G}_r$. As before, we may assume that $\mathbf{L} = \mathbf{L}_1$ is a non-trivial normal $\mathbb{Q}$-subgroup of $\mathbf{G}$, up to replacing $\mathbf{H}$ with $\mathbf{H}/\mathbf{L}_2 = \mathbf{G} \times (\mathbf{B}/\mathbf{L}_2)$ and using Proposition A.7. By Corollary 2.3, $\mathbf{L}_1$ is an almost-direct product of some of the $\mathbb{Q}$-almost simple factors of $\mathbf{G}$. If $\mathbf{L}_1 = \mathbf{G}$, the argument in the $\mathbb{Q}$-almost simple applies unaffectedly; upon rearrengement of the factors, we may thus assume that $\mathbf{L}_1 = \mathbf{G}_{s+1} \cdots \mathbf{G}_r$ for some $1 \leq s \leq r-1$, and denote $\mathbf{G}' = \mathbf{G}_1 \cdots \mathbf{G}_s$. Let $G_L = G \cap \mathbf{L}_1(\mathbb{Q}_S), \Gamma_L = \Gamma \cap G_L, X_L = \Gamma_L \backslash G_L$; also, set G' to be the projection of G onto $\mathbf{G}'(\mathbb{Q}_S)$, Γ' the image of Γ under the same projection map, and $X' = \Gamma' \backslash G'$.

The projection $\tilde{\mu}$ of μ to $X' \times Y$ is a $\mathbb{Z}^d$-invariant and ergodic joining of m_Y and the Haar measure $m_{X'}$ on X', where $\mathbb{Z}^d$ acts on $X' \times Y$ via the projected homomorphism $\mathbb{Z}^d \xrightarrow{a} G \times B \to G' \times B$, which we denote $\tilde{a}$. It is clear that $\tilde{\mu}$ and $\tilde{a}$ fulfill the assumptions of Theorem 1.3; as X' is an S-arithmetic quotient of the group $\mathbf{G}'$, having $s \leq r-1$ $\mathbb{Q}$-almost simple factors, the induction hypothesis permits to deduce that $\tilde{\mu} = m_{X'} \times m_Y$. Therefore, μ is a measure on the product $(X' \times Y) \times X_L$, projecting to $m_{X'} \times m_Y$ onto $X' \times Y$ and invariant under the finite-index subgroup $L_1 = L \cap G_L < G_L$. Just as in the case of a $\mathbb{Q}$-almost simple group $\mathbf{G}$, define $A_{L_1} = \{(a_1', a_1^L, a_2) \in A < G' \times G_L \times B : a_1^L \in L_1\}$, and consider an $A_{L_1}L_1$-ergodic decomposition of μ. Arguing as before, it can be inferred that L_1 acts ergodically on X_L with respect to m_{X_L}, and Proposition 8.7 delivers once more $\mu = (m_{X'} \times m_Y) \times m_{X_L}$, that is, rearranging factors, $\mu = m_X \times m_Y$.

Remark 8.8. For the sake of emphasizing the gist of the above argument, we are tacitly assuming that $\mathbf{G}$ is a direct product of its $\mathbb{Q}$-almost simple factors and X splits neatly as the product $X_L \times X'$. This allows for a direct application of Proposition 8.7. If $\mathbf{G}$ is not simply connected, Proposition A.7 provides an adequate replacement: combining L-invariance of μ with the fact that its projection $\tilde{\mu}$ modulo L equals $m_{X'} \times m_Y$, it follows all the same that $\mu = m_X \times m_Y$.

8.2. **The case $\mathbf{G}$ perfect.** The case of a semisimple $\mathbf{G}$ being established, it is rather straightfoward to deduce the result when $\mathbf{G}$ is only assumed to be perfect. It essentially follows from the Levi decomposition of a linear algebraic group, together with a classification result analogous to [17, Thm. 1.6] and phrased in Proposition 8.10.

Assume thus that $\mathbf{G}$ is a Zariski-connected, perfect $\mathbb{Q}$-group.

Lemma 8.9. *Let $\mathbf{G} = \mathbf{G}_{ss} \ltimes R_{\mathrm{u}}(\mathbf{G})$ be a Levi decomposition of $\mathbf{G}$.*

(1) The Levi factor $\mathbf{G}_{ss}$ is semisimple.
(2) If $A_1 < \mathbf{G}(\mathbb{Q}_S)$ is a diagonalizable subgroup, then A_1 is contained in $\mathbf{G}_{ss}(\mathbb{Q}_S)$.

Proof. For the first assertion, observe that $\mathbf{G} = [\mathbf{G}, \mathbf{G}] < [\mathbf{G}_{ss}, \mathbf{G}_{ss}] \ltimes R_{\mathrm{u}}(\mathbf{G})$, whence $\mathbf{G}_{ss}$ is necessarily equal to its commutator $[\mathbf{G}_{ss}, \mathbf{G}_{ss}]$,which is semisimple (cf. Thm. 2.1).

As for the second statement, let $a = (a_\sigma)_{\sigma\in S}$ be an element of A_1; it suffices to show that $a_\sigma \in \mathbf{G}_{ss}(\mathbb{Q}_\sigma)$ for every $\sigma \in S$. To this end, we work inside the groups of $\overline{\mathbb{Q}_\sigma}$-points of $\mathbf{G}, \mathbf{G}_{ss}$ and $R_{\mathrm{u}}(\mathbf{G})$, $\overline{\mathbb{Q}_\sigma}$ designating an algebraic closure of the field $\mathbb{Q}_\sigma$. Since $\mathbf{G}(\mathbb{Q}_\sigma) = \mathbf{G}_{ss}(\mathbb{Q}_\sigma) \ltimes R_{\mathrm{u}}(\mathbf{G})(\mathbb{Q}_\sigma)$, we have $a_\sigma = a_\sigma^{(G_{ss})} a_\sigma^{(U)}$, for some $a_\sigma^{(G_{ss})} \in \mathbf{G}_{ss}(\mathbb{Q}_\sigma)$, $a_\sigma^{(U)} \in R_{\mathrm{u}}(\mathbf{G})(\mathbb{Q}_\sigma)$. The canonical projection $\mathbf{G}(\overline{\mathbb{Q}_\sigma}) \to \mathbf{G}_{ss}(\overline{\mathbb{Q}_\sigma})$, with kernel $R_u(\mathbf{G})(\overline{\mathbb{Q}_\sigma})$, is a homomorphism of algebraic groups, and as such it preserves the Jordan decomposition of elements ([65, Thm. 2.4.8]). Because a_σ is diagonalizable by assumption, it follows that $a_\sigma^{(G_{ss})}$ is a diagonalizable element in $\mathbf{G}_{ss}(\overline{\mathbb{Q}_\sigma})$, thus *a fortiori* in $\mathbf{G}(\overline{\mathbb{Q}_\sigma})$. Uniqueness of the Jordan decomposition of an element in $\mathbf{G}(\overline{\mathbb{Q}_\sigma})$ now forces $a_\sigma^{(U)} = e$, the identity element of the group $\mathbf{G}(\overline{\mathbb{Q}_\sigma})$; hence, $a_\sigma = a_\sigma^{(G_{ss})} \in \mathbf{G}_{ss}(\mathbb{Q}_\sigma)$, as claimed in the statement. □

Fix thus a Levi decomposition $\mathbf{G} = \mathbf{G}_{ss} \ltimes R_{\mathrm{u}}(\mathbf{G})$ of $\mathbf{G}$. As a consequence of the proposition, the image of the class-$\mathcal{A}'$ homomorphism $a_G \colon \mathbb{Z}^d \to G$ is contained in $G \cap \mathbf{G}_{ss}(\mathbb{Q}_S)$. Denote by G_{ss} the finite-index subgroup of $\mathbf{G}_{ss}(\mathbb{Q}_S)$ given by the image of G under the canonical projection $\mathbf{G}(\mathbb{Q}_S) \to \mathbf{G}_{ss}(\mathbb{Q}_S)$; since $R_{\mathrm{u}}(\mathbf{G})(\mathbb{Q}_S)$ has no proper finite-index subgroups, G contains $R_{\mathrm{u}}(\mathbf{G})(\mathbb{Q}_S)$, so that $G = G_{ss} \ltimes R_{\mathrm{u}}(\mathbf{G})(\mathbb{Q}_S)$. Furthermore, on the level of algebraic groups, the projection $\mathbf{G} \to \mathbf{G}_{ss}$ is defined over $\mathbb{Q}$, hence (see [40, Lem. 3.1.3]) the image Γ_{ss} of Γ under the projection $G \to G_{ss}$ is an S-arithmetic subgroup embedding diagonally as a discrete subgroup of G_{ss}; pushing forward the Haar-Siegel measure m_X on X via the map $\Gamma\backslash G \to \Gamma_{ss}\backslash G_{ss}$ yields a G_{ss}-invariant probability measure $m_{X_{ss}}$ on $X_{ss} = \Gamma_{ss}\backslash G_{ss}$.

Now let μ be a $\mathbb{Z}^d$-invariant and ergodic joining of m_X and m_Y, as in the assumptions of Theorem 1.3. The push-foward $\pi_*\mu$ of μ under the projection map $\pi\colon X \times Y \to X_{ss} \times Y$ is a $\mathbb{Z}^d$-invariant and ergodic joining of $m_{X_{ss}}$ and m_Y, where $\mathbb{Z}^d$ acts on the product $X_{ss} \times Y$ via the projection of the homomorphism $a_G \times a_B \colon \mathbb{Z}^d \to G \times B$ to $G_{ss} \times B$. As X_{ss} is an S-arithmetic quotient of a semisimple $\mathbb{Q}$-group, we know from Section 8.1 that $\pi_*\mu = m_{X_s} \times m_Y$. The following result allows to conclude that $\mu = m_X \times m_Y$.

Proposition 8.10 (cf. [17, Thm. 1.6])**.** *Let $\mathbf{G} = \mathbf{G}_{ss} \ltimes R_{\mathrm{u}}(\mathbf{G}), G_{ss}, \Gamma_{ss}$ and X_{ss} be as above, $a_G \colon \mathbb{Z}^d \to G, a_B \colon \mathbb{Z}^d \to B$ homomorphisms satisfying the hypotheses of Theorem 1.3. Suppose that μ is invariant and ergodic under the diagonal action of $\mathbb{Z}^d$ on $X \times Y$, and projects to the product measure $m_{X_{ss}} \times m_Y$ on $X_{ss} \times Y$. Then $\mu = m_X \times m_Y$.*

Throughout the proof, we tacitly make use of the following observation: for a given probability measure μ on $X \times Y$, the property of projecting to the Haar-Siegel measure on $X_{ss} \times Y$ does not depend on the choice of the Levi factor $\mathbf{G}_{ss}$ of $\mathbf{G}$, since any two different Levi factors are conjugated inside $\mathbf{G}$ by an element of $R_{\mathrm{u}}(\mathbf{G})(\mathbb{Q})$ (cf. [53, Thm. 2.3]).

Proof. We argue by induction on the number of $\mathbb{Q}$-simple factors of the Levi factor $\mathbf{G}_{ss}$. Recall that, by the argument at the beginning of Section 8, μ is additionally invariant under a finite-index subgroup $L < \mathbf{L}(\mathbb{Q}_S)$, where $\mathbf{L}$ is a connected, normal $\mathbb{Q}$-subgroup of $\mathbf{G} \times \mathbf{B}$ of class $\mathcal{F}$.

If $\mathbf{L} = \mathbf{L}_{ss} \ltimes R_{\mathrm{u}}(\mathbf{L})$ is a Levi decomposition of $\mathbf{L}$, then $\mathbf{L}_{ss}$ is semisimple and contained in $\mathbf{G}$, and $R_{\mathrm{u}}(\mathbf{L})$ is contained in $R_{\mathrm{u}}(\mathbf{G} \times \mathbf{B}) = R_{\mathrm{u}}(\mathbf{G}) \times R_{\mathrm{u}}(\mathbf{B})$ (cf. the proof of Proposition 8.4 in Appendix B). By means of this decomposition, and using that the subgroup $\mathbf{L}$ is normal in $\mathbf{G} \times \mathbf{B}$, it is straightforward to realize that $\mathbf{L}_{ss}$ is also a normal subgroup of $\mathbf{G} \times \mathbf{B}$. Also, the semisimple group $\mathbf{L}_{ss}$ is contained in a Levi factor of $\mathbf{G}$. Combining this with the fact that $R_{\mathrm{u}}(\mathbf{G}\times\mathbf{B})\cap\mathbf{L}_{ss} = R_{\mathrm{u}}(\mathbf{L}_{ss})$ is trivial[19], which in turn implies that $\mathbf{L}_{ss}$ and $R_{\mathrm{u}}(\mathbf{G}\times\mathbf{B})$ commute, it follows that there exists a semisimple $\mathbb{Q}$-group $\mathbf{G}'_{ss} < \mathbf{G}$ such that $\mathbf{G} = \mathbf{L}_{ss} \times (\mathbf{G}'_{ss} \ltimes R_{\mathrm{u}}(\mathbf{G})) \times \mathbf{B}$; for notational simplicity, denote by $\mathbf{G}'$ the perfect $\mathbb{Q}$-group $\mathbf{G}'_{ss} \ltimes R_{\mathrm{u}}(\mathbf{G})$. Corresponding to such a decomposition on the level of algebraic groups, there is a decomposition of the homogeneous space $X \times Y$ as a product $X_L \times X' \times Y$, where $X_L = (\Gamma \cap \mathbf{L}_{ss}(\mathbb{Q}_S))\backslash(G \cap \mathbf{L}_{ss}(\mathbb{Q}_S))$ and $X' = (\Gamma \cap \mathbf{G}'(\mathbb{Q}_S))\backslash(G \cap \mathbf{G}'(\mathbb{Q}_S))$.

If the normal subgroup $R_{\mathrm{u}}(\mathbf{L}) < \mathbf{G}\times\mathbf{B}$ is non-trivial, we can turn our attention to the projection $\bar{\mu}$ of μ onto the quotient $q(\Delta)\backslash(G\times B)/R_{\mathrm{u}}(\mathbf{L})(\mathbb{Q}_S)$, where $q\colon G\times B \to (G\times B)/R_{\mathrm{u}}(\mathbf{L})(\mathbb{Q}_S)$ is the canonical projection. If $\bar{\mu}$ satisfies the conclusion of the proposition, then so does μ, as explained in Appendix A.3; to this end, notice that $R_{\mathrm{u}}(\mathbf{L})(\mathbb{Q}_S)$ acts with compact orbits on $X \times Y$ (cf. footnote 16), so that in particular Proposition A.7 applies. Therefore, we might assume that $R_{\mathrm{u}}(\mathbf{L})$ is trivial, so that $\mathbf{L} = \mathbf{L}_{ss}$ is non-trivial. In this case, the Levi factor $\mathbf{G}'_{ss}$ of the perfect group $\mathbf{G}'$ has fewer $\mathbb{Q}$-simple factors with respect to $\mathbf{G}_{ss}$. If μ' stands for the projection of μ to $X' \times Y$, then all the assumptions of the proposition are satisfied, with $\mathbf{G}$ replaced by $\mathbf{G}'$, m_X by $m_{X'}$ and a_G by its projection onto $\mathbf{G}'(\mathbb{Q}_S)$; the induction hypothesis thus yields $\mu' = m_{X'} \times m_Y$. Finally, since μ is invariant under a finite-index subgroup of $\mathbf{L}_{ss}(\mathbb{Q}_S)$, we may again combine Proposition 8.7 with an ergodic-decomposition argument analogous to the one in Section 8.1 to obtain that $\mu = m_{X_L} \times \mu'$, where m_{X_L} is the Haar measure on X_L. Hence, we get $\mu = m_{X_L} \times m_{X'} \times m_Y = m_X \times m_Y$, which finishes the proof. □

The proof of Theorem 1.3 is concluded.

Appendix A. Some properties of lattices and measures on homogeneous spaces

We collect here various, mostly well-known statements, interspersed in the main body of the manuscript and pertaining to lattices in locally compact groups and algebraic measures on homogeneous spaces.

Keeping with our usual terminology, a locally compact group is intended to be a Hausdorff, locally compact second countable topological group.

A.1. **Projection of lattices.**

Lemma A.1. *Let G be a locally compact group, $\Gamma < G$ a lattice, $L, R < G$ closed subgroups with R normal in G. Assume that the following two conditions hold:*

(1) the image $\Lambda \subset G/R$ of Γ under the natural projection $q\colon G \to G/R$ is a lattice in G/R;
(2) the induced map $\pi\colon \Gamma\backslash G \to \Lambda\backslash(G/R)$ restricts to a surjective map on $\Gamma L = \{\Gamma g \in \Gamma\backslash G : g \in L\}$.

The the image $q(L)$ has non-empty interior in G/R.

Proof. Missing. □

[19]In characteristic zero, every unipotent algebraic group is Zariski-connected, thus $R_{\mathrm{u}}(\mathbf{G} \times \mathbf{B}) \cap \mathbf{L}_{ss}$ is connected. Therefore, the latter group is a connected normal unipotent subgroup of $\mathbf{L}_{ss}$, whence $R_{\mathrm{u}}(\mathbf{L}_{ss}) \supset R_{\mathrm{u}}(\mathbf{G} \times \mathbf{B}) \cap \mathbf{L}_{ss}$. On the other hand, $R_{\mathrm{u}}(\mathbf{L}_{ss})$ is invariant under all automorphisms of $\mathbf{L}_{ss}$, hence in particular under the restriction of each inner automorphism of $\mathbf{G} \times \mathbf{B}$ to $\mathbf{L}_{ss}$; it follows that $R_{\mathrm{u}}(\mathbf{L}_{ss})$ is a connected, normal unipotent subgroup of $\mathbf{G} \times \mathbf{B}$, so that $R_{\mathrm{u}}(\mathbf{L}_{ss}) \subset R_{\mathrm{u}}(\mathbf{G} \times \mathbf{B}) \cap \mathbf{L}_{ss}$.

Lemma A.2. *Let G be a locally compact group, $R < G$ a closed, normal subgroup, $\Gamma < G$ a discrete subgroup. Let $q\colon G \to G/R$ denote the canonical projection, $\Lambda = q(\Gamma)$. Then*

$$\Lambda\backslash(G/R) \simeq \Gamma\backslash G/R$$

as topological G/R-spaces.

Proof. We start by defining a map

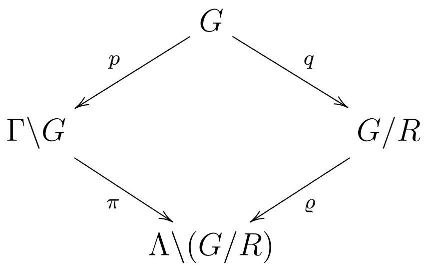
$$\pi : \Gamma\backslash G \to \Lambda\backslash(G/R)$$
$$\pi(\Gamma g) = \Lambda(gR).$$

In order to see that this is well-defined, we first note that for any two $g, \tilde{g} \in G$ we have

$$\Lambda(gR) = \Lambda(\tilde{g}R) \iff \exists\gamma \in \Gamma\, \gamma gR = \tilde{g}R.$$

Indeed, the left-hand side implies that there is some $\gamma \in \Gamma$ such that $(\gamma R)(gR) = \tilde{g}R$, and thus in particular $\gamma gR = \tilde{g}R$. The opposite direction follows similarly. Hence, if $g, \tilde{g} \in G$ satisfy $\gamma g = \tilde{g}$ for some $\gamma \in \Gamma$, then $\pi(\Gamma g) = \pi(\Gamma\tilde{g})$ as required.

To show that π is continuous, consider the diagram



(A.1)

and note that p is a local homeomorphism. Since locally we have $\pi = \varrho \circ q \circ p^{-1}$, we obtain that π is continuous.

Finally, we observe that the fibers of π are precisely the R-orbits in $\Gamma\backslash G$, that is, $\pi(\Gamma g) = \pi(\Gamma\tilde{g})$ for $g, \tilde{g} \in G$ if and only if there is some $r \in R$ such that $\Gamma gr = \Gamma\tilde{g}$. To this end, assume that $\pi(\Gamma g) = \pi(\Gamma\tilde{g})$; as argued above, this is equivalent to the existence of $\gamma \in \Gamma$ such that $\gamma gR = \tilde{g}R$. Thus there is some $r \in R$ such that $\gamma gr = \tilde{g}$; it follows that $\Gamma\tilde{g} = \Gamma gr \subseteq \Gamma gR$ and the fibers of π are therefore contained in R orbits. The fact that π is constant on R-orbits is immediate. ☐

Proposition A.3. *Let G be a locally compact group, $\Gamma < G$ a lattice in G, $K < G$ a compact normal subgroup. Denote by $\pi\colon G \to G/K$ the canonical projection map. Then $\pi(\Gamma)$ is a lattice in G/K.*

Proof. We begin by showing that $\pi(K)$ is discrete in G/K. Let $\gamma_n \in \Gamma$ be such that the sequence $(\pi(\gamma_n))_n$ converges towards the identity in G/K. This implies that there exists a sequence $(k_n)_n$ of elements of K such that $\gamma_n k_n \to e_G$ in G as n goes to infinity. Upon replacing $(k_n)_n$ by a converging subsequence (using that K is compact and G is metrizable), we may assume that $k_n \to k \in K$. Thus, $\gamma_n \to k^{-1} \in K \cap \Gamma$; discreteness of Γ forces $\gamma_n = k^{-1}$ for all n sufficiently large, which shows that $\pi(\gamma_n)$ is the identity $e_{G/K}$ in G/K for all such n. This argument shows that $e_{G/K}$ is an isolated point in $\pi(\Gamma)$, thereby proving that the latter is a discrete subgroup.

It remains to prove that the space of right cosets $\pi(\Gamma)\backslash(G/K)$ admits a G/K-invariant Borel probability measure. As shown in Lemma A.2, $\pi(\Gamma)\backslash(G/K) \simeq \Gamma\backslash G/K$ as G/K-spaces. Denote by $\rho\colon \Gamma\backslash G \to \Gamma\backslash G/K$ the canonical projection; then ρ is a proper, open map. Indeed, for every open set $V \subset \Gamma\backslash G$, the set $\rho^{-1}(\rho(V)) = VK$ is open as union of translates of V, whence $\rho(V)$ is open, since $\Gamma\backslash G/K$ carries the quotient topology defined by the map ρ. To prove that ρ is proper, let $L \subset \Gamma\backslash G/K$ be a compact subset; we claim that there is a compact subset $M \subset \Gamma\backslash G$ such that $L = \rho(M)$. Indeed, choose for every $x \in L$ a compact subset $V_x \subset \Gamma\backslash G$ so that $\rho(V_x)$ is a neighborhood of x. By compactness of L, there is a finite subset $\mathcal{X} \subset \Gamma\backslash G$ such that

$$L \subset \bigcup_{x\in\mathcal{X}} \rho(V_x)^\circ ,$$

and thus the set $M = \rho^{-1}(L) \cap \left(\bigcup_{x\in\mathcal{X}} V_x\right)$ has the desired properties. This implies that $\rho^{-1}(L) = MK$ is compact, proving that ρ is proper.

As a consequence, for every continuous, compactly supported function $\psi\colon \Gamma\backslash G/K \to \mathbb{C}$, the function $\psi \circ \rho$ is continuous and compactly supported on $\Gamma\backslash G$. We can thus define the linear functional $\mu\colon \mathrm{C_c}(\Gamma\backslash G/K) \to \mathbb{C}$, which to each $\psi \in \mathrm{C_c}(\Gamma\backslash G/K)$ assigns the value

$$\mu(\psi) = \int_{\Gamma\backslash G} (\psi \circ \rho)\, \mathrm{d}m_{\Gamma\backslash G} ,$$

where $m_{\Gamma\backslash G}$ denotes the G-invariant Borel probability measure on $\Gamma\backslash G$. The functional μ is clearly positive, in the sense that $\psi \geq 0$ implies $\mu(\psi) \geq 0$, and bounded in operator norm by 1; Riesz representation theorem thus gives a Borel probability measure, again denoted by μ, such that

$$\mu(\psi) = \int_{\Gamma\backslash G/K} \psi\, \mathrm{d}\mu \quad \text{for all } \psi \in \mathrm{C_c}(\Gamma\backslash G/K).$$

We are left to show that μ is G/K-invariant. Fix $h \in G$ and denote by $\psi_{hK} \in \mathrm{C_c}(\Gamma\backslash G/K)$ the function defined by

$$\psi_{hK}(\Gamma gK) = \psi(\Gamma ghK) \ , \ g \in G.$$

Similarly, if $f \in \mathrm{C_c}(\Gamma\backslash G)$, denote by f_h the function defined by $f_h(\Gamma g) = f(\Gamma gh)$ for all $g \in G$. Using this notation, we have

$$(\psi_{hK} \circ \rho)(\Gamma g) = \psi(\Gamma gh) = (\psi \circ \rho)_h(\Gamma g) \text{ for all } g \in G,$$

so that

$$\mu(\psi_{hK}) = m_{\Gamma\backslash G}((\psi \circ \rho)_h) = m_{\Gamma\backslash G}(\psi \circ \rho) = \mu(\psi) ,$$

where the equality in the middle follows by G-invariance of $m_{\Gamma\backslash G}$. The proof is concluded. □

A.2. **Haar measure on finite volume orbits.** Let G be a locally compact group, $\Gamma < G$ a lattice, $X = \Gamma\backslash G$, $H < G$ a closed subgroup, acting on X by right translation. Fix a point $x = \Gamma g \in X$ and define

$$\mathrm{Stab}_H(x) = \{h \in H : h \cdot x = x\}.$$

There exists a continuous H-equivariant map

$$H/\mathrm{Stab}_H(x) \ni h\, \mathrm{Stab}_H(x) \mapsto h \cdot x \in X, \tag{A.2}$$

which is a bijection onto the H-orbit of x

$$H \cdot x = \{h \cdot x : h \in H\}.$$

An easy computation shows that $\mathrm{Stab}_H(x) = H \cap g^{-1}\Gamma g$, which is a discrete subgroup of H. If $\Lambda = H \cap g^{-1}\Gamma g$ is a lattice in H, then the H-orbit $H \cdot x$ is closed in X (see [54, Thm. 1.13]), the map in (A.2) is an homeomorphism onto $H \cdot x$, and pushing forward the unique H-invariant probability measure $m_{H/\Lambda}$ via this map yields an H-invariant Borel probability measure on $H \cdot x$, called the *Haar measure* on the H-orbit of x. A *periodic H-orbit* is an H-orbit carrying an H-invariant Haar measure. Haar measures on orbits of intermediate subgroups are referred to as *algebraic* or *homogeneous* measures.

A.3. **Invariant lifts of algebraic measures.** This subsection characterizes measures on homogeneous spaces which are invariant under a normal subgroup and whose projection to the corresponding quotient is an algebraic measure. This is used to reduce the proof of Theorem 1.1 to the case where there is some non-trivial unipotent invariance stemming from the semisimple side (cf. Section 8.1), as well as to deduce algebraicity of joinings in the perfect case (cf. Section 8.2). Let G be a locally compact group with identity element e, $\Gamma < G$ a discrete subgroup. We assume furthermore that $R < G$ is a closed normal subgroup such that all R-orbits on the homogeneous space $\Gamma\backslash G$ are periodic; since R is normal, it is equivalent to require the property to hold for a single R-orbit, as any other orbit is then obtained by translation. For any closed subgroup $L < G$ and any periodic L-orbit $L \cdot x \subset \Gamma\backslash G$, we indicate with $m_{L\cdot x}$ the unique L-invariant measure carried by $L \cdot x$. Denote by $q\colon G \to G/R$ the canonical projection and suppose that $\bar{\Gamma} \coloneqq q(\Gamma) < G/R$ is discrete. It will be convenient to set $\bar{G} \coloneqq G/R$. We let $\pi\colon \Gamma\backslash G \to \Gamma\backslash G/R$ denote the projection as in Lemma A.2.

We start with a lemma concerning the topology of $\Gamma\backslash G/R$:

Lemma A.4. *For G, Γ, R as above, the double coset space $\Gamma\backslash G/R$ is Hausdorff, locally compact and second countable.*

Proof. It follows at once from the identification in Lemma A.2 and the analogous properties for a quotient $\Lambda\backslash H$, where $\Gamma < H$ is a closed subgroup of a locally compact group H. □

Consequently, the double coset space $\Gamma\backslash G/R$ is as nice a space as required to apply Riesz's representation theorem and similar tools.

Next, we introduce the averaging operator over periodic R-orbits; this is instrumental in formulating an analogue of Weil's classical folding-unfolding formula (cf. [59, §3.2]) on $\Gamma\backslash G$.

Lemma A.5. *For every $\psi \in \mathrm{C_c}(\Gamma\backslash G/R)$, there exists $\varphi \in \mathrm{C_c}(\Gamma\backslash G)$ such that*

$$\psi\big(\pi(x)\big) = \int_{R\cdot x} \varphi(y)\mathrm{d}m_{R\cdot x}(y) \quad \text{for every } x \in \Gamma\backslash G.$$

More precisely, the linear operator $T\colon \mathrm{C_c}(\Gamma\backslash G) \to \mathrm{C_c}(\Gamma\backslash G/R)$ defined by

$$T\varphi\big(\pi(x)\big) = \int_{R\cdot x} \varphi(y)\mathrm{d}m_{R\cdot x}(y)$$

is positive, that is $T\varphi \geq 0$ whenever $\varphi \geq 0$, and surjective.

Proof. We first prove that $I\colon \mathrm{C_c}(\Gamma\backslash G) \to \mathrm{C}(\Gamma\backslash G)$ defined by

$$I\varphi(x) \coloneqq \int_{\Gamma\backslash G} \varphi(y)\mathrm{d}m_{R\cdot x}(y)$$

is well-defined and the image is constant on R-orbits. First we check that $I\varphi$ is continuous. We may argue with sequences, as $\Gamma\backslash G$ satisfies the first axiom of countability. Let thus $(x_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence in $\Gamma\backslash G$ such that $x_\infty = \lim_{n\to\infty} x_n$. Fix $(\varepsilon_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that $\varepsilon_n \to e$ and[20] $x_n = x_\infty\varepsilon_n$. We have

$$\begin{aligned}|I\varphi(x_n) - I\varphi(x_\infty)| &= \left|\int_{\Gamma\backslash G} \varphi(y)\mathrm{d}m_{R\cdot x_n}(y) - I\varphi(x_\infty)\right| = \left|\int_{\Gamma\backslash G} \varphi(y)\mathrm{d}(\varepsilon_n)_*m_{R\cdot x_\infty}(y) - I\varphi(x_\infty)\right| \\ &= \left|\int_{\Gamma\backslash G} \varphi(y\varepsilon_n)\mathrm{d}m_{R\cdot x_\infty}(y) - I\varphi(x_\infty)\right| = \left|\int_{\Gamma\backslash G} \big((\varepsilon_n\cdot\varphi) - \varphi\big)(y)\mathrm{d}m_{R\cdot x_\infty}(y)\right| \\ &\leq \|\varepsilon_n\cdot\varphi - \varphi\|_\infty \overset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

[20]The existence of such a sequence $(\varepsilon_n)_n$ is ensured by the well-known fact that the projection $G \to \Gamma\backslash G$ is a local homeomorphism.

by strong continuity of the Koopman representation of G on $\mathrm{C_c}(\Gamma\backslash G)$ with respect to the supremum norm; this in turn is a consequence of compactly supported continuous functions on $\Gamma\backslash G$ being uniformly continuous.

It follows that there is a unique $f \in \mathrm{C}(\Gamma\backslash G/R)$ such that $I\varphi = f \circ \pi$. We define $T\varphi := f$. Linearity and positivity of T are immediate from the definition. To show that $T\varphi$ has compact support for any $\varphi \in \mathrm{C_c}(\Gamma\backslash G)$, we prove that $T\varphi$ vanishes outside of $\pi(\operatorname{supp}\varphi)$. Let $x \in \Gamma\backslash G$ such that $\pi(x) \in (\Gamma\backslash G/R) \setminus \pi(\operatorname{supp}\varphi)$. Then $r \cdot x \notin \operatorname{supp}\varphi$ for any $r \in R$. Hence $\varphi(r \cdot x) = 0$ for all $r \in R$, and thus $T\varphi\big(\pi(x)\big) = I\varphi(x) = 0$. This shows that $\operatorname{supp} T\varphi \subseteq \pi(\operatorname{supp}\varphi)$ and therefore $T\varphi \in \mathrm{C_c}(\Gamma\backslash G/R)$.

It remains to show that T is surjective. Assume that $\psi \in \mathrm{C_c}(\Gamma\backslash G/R)$. Let $C \subseteq G$ a compact neighbourhood of $e \in G$. As $\operatorname{supp}\psi$ is compact and π is an open map, there exist $x_1, \dots, x_n \in \pi^{-1}(\operatorname{supp}\psi)$ such that $\{\pi(x_i C) : 1 \leq i \leq n\}$ covers $\operatorname{supp}\psi$. Define

$$K := \bigcup_{1\leq i\leq n} \big(\pi^{-1}(\operatorname{supp}\psi) \cap x_i C\big) \ ,$$

which is a compact subset of $\Gamma\backslash G$. It is clear that $\pi(K) = \operatorname{supp}\psi$. By means of Urysohn's lemma, choose $f \in \mathrm{C_c}(\Gamma\backslash G)$ non-negative such that $f\big|_K > 0$. Let $\xi \in \operatorname{supp}\psi$ arbitrary. By construction, there is some $x \in K$ such that $\pi(x) = \xi$. As f is continuous and $f(x) > 0$, there is a neighbourhood $V_x \subseteq R$ containing e for which the map $V_x \ni r \mapsto r \cdot x \in R \cdot x$ is a homeomorphism and such that $f|_{V_x \cdot x} > 0$. Hence by positivity of f we get

$$Tf(\pi(x)) = \int_{\Gamma\backslash G} f(y)\mathrm{d}m_{R\cdot x}(y) \geq \int_{V_x} f(r\cdot x)\mathrm{d}r > 0,$$

which shows that $Tf|_{\operatorname{supp}\psi} > 0$. Hence we define

$$\varphi(x) := \begin{cases} (\psi\circ\pi)(x)\frac{f(x)}{Tf(\pi(x))} & \text{if } x \in \pi^{-1}(\operatorname{supp}\psi), \\ 0 & \text{else.} \end{cases}$$

It holds that $\operatorname{supp}\varphi \subseteq \operatorname{supp} f$, whence $\operatorname{supp}\varphi$ is compact. For $x \in \pi^{-1}(\operatorname{supp}\psi)$ we calculate

$$\begin{aligned} T\varphi\big(\pi(x)\big) &= \int_{\Gamma\backslash G} (\psi\circ\pi)(y)\frac{f(y)}{(Tf\circ\pi)(y)}\mathrm{d}m_{R\cdot x}(y) \\ &= \int_{\Gamma\backslash G} (\psi\circ\pi)(x)\frac{f(y)}{(Tf\circ\pi)(x)}\mathrm{d}m_{R\cdot x}(y) \\ &= \frac{\psi\big(\pi(x)\big)}{Tf\big(\pi(x)\big)}\int_{\Gamma\backslash G} f(y)\mathrm{d}m_{R\cdot x}(y) = \psi\big(\pi(x)\big) \end{aligned}$$

and thus $T\varphi = \psi$ as desired. ☐

The following lemma elucidates the relationship between algebraic measures on the two spaces $\Gamma\backslash G$ and $\Gamma\backslash G/R$.

Lemma A.6. *Let $\bar{L} < G/R$ a closed subgroup, $g \in G$ an element such that the orbit $\bar{L}\cdot\bar{\Gamma}q(g)$ is periodic. Set $L = q^{-1}(\bar{L})$. Then $\pi(L\cdot\Gamma g) = \bar{L}\cdot\bar{\Gamma}q(g)$, the orbit $L\cdot\Gamma g$ is periodic, and $m_{\bar{L}\cdot\bar{\Gamma}q(g)} = \pi_* m_{L\cdot\Gamma g}$.*

Proof. The first part of the statement follows from the definition of π (cf. Lemma A.2) and the fact that $\bar{L} = q(L)$. For the last two assertions, we will assume without loss of generality that $G = L$, in order to simplify notation, so that we are left to show that $\Gamma\backslash G$ admits a G-invariant probability measure, given that $\bar{\Gamma}\backslash\bar{G}$ has a $\bar{G}$-invariant probability measure $m_{\bar{\Gamma}\backslash\bar{G}}$. As before, let $m_{R\cdot x}$ be the unique R-invariant probability measure carried by the R-orbit $R\cdot x$. Notice that, for any $g \in G$ and $x \in X$, it holds that $m_{R\cdot(g\cdot x)} = g_* m_{R\cdot x}$, so that in particular the map

$x \mapsto m_{R\cdot x}$ is continuous and constant on R-orbits. Let $T\colon \mathrm{C_c}(\Gamma\backslash G) \to \mathrm{C_c}(\Gamma\backslash G/R)$ be the linear operator

$$T\varphi\big(\pi(x)\big) = \int_{\Gamma\backslash G} \varphi(y)\mathrm{d}m_{R\cdot x}(y) \quad (x \in \Gamma\backslash G)$$

defined in Lemma A.5. Define a positive linear functional I on $\mathrm{C_c}(\Gamma\backslash G)$ by

$$I\varphi = \int_{\bar\Gamma\backslash\bar G} T\varphi\, dm_{\bar\Gamma\backslash\bar G} \quad \forall\varphi \in \mathrm{C_c}(\Gamma\backslash G).$$

This corresponds to a Borel probability measure $m_{\Gamma\backslash G}$ on $\Gamma\backslash G$, which we claim is invariant under G. Let $g \in G$ and $\varphi \in \mathrm{C_c}(\Gamma\backslash G)$, and extend the G-action to $\mathrm{C_c}(\Gamma\backslash G)$ by setting $(g\cdot\varphi)(x) = \varphi(g^{-1}\cdot x), x \in \Gamma\backslash G$. We have

$$\begin{aligned}T\varphi\big(q(g)\cdot\pi(x)\big) = T\varphi\big(\pi(g\cdot x)\big) &= \int_{\Gamma\backslash G}\varphi(y)\mathrm{d}m_{R\cdot(g\cdot x)}(y) = \int_{\Gamma\backslash G}\varphi(y)\mathrm{d}g_* m_{R\cdot x}(y)\\ &= \int_{\Gamma\backslash G}\varphi(g\cdot y)\mathrm{d}m_{R\cdot x}(y) = \int_{\Gamma\backslash G}(g^{-1}\cdot\varphi)(y)\mathrm{d}m_{R\cdot x}(y) = T(g^{-1}\cdot\varphi)\big(\pi(x)\big),\end{aligned}$$

that is, $q(g)\cdot T\varphi = T(g\cdot\varphi)$.It follows that

$$m_{\Gamma\backslash G}(g\cdot\varphi) = m_{\bar\Gamma\backslash\bar G}\big(T(g\cdot\varphi)\big) = m_{\bar\Gamma\backslash\bar G}(T\varphi) = m_{\Gamma\backslash G}(\varphi),$$

which ends the proof.

□

As a last preliminary step towards the proof of the announced result, notice that in the last part of the foregoing proof we have shown the natural extension of Weil's folding-unfolding formula to our setup:

$$\int_{\Gamma\backslash G}\varphi\;\mathrm{d}m_{\Gamma\backslash G} = \int_{\Gamma\backslash G/R} T\varphi\;\mathrm{d}m_{\Gamma\backslash G/R} \tag{A.3}$$

for every $\varphi \in \mathrm{C_c}(\Gamma\backslash G)$.

Finally, we are in a position to show:

Proposition A.7. *Let μ be an R-invariant probability measure on $\Gamma\backslash G$ and assume that $\pi_*\mu$ is algebraic, that is, there exist a closed subgroup $\bar L < G/R$ and an element $\bar g \in G/R$ such that $\pi_*\mu$ is the unique $\bar L$-invariant measure supported on the orbit $\bar L\cdot\bar\Gamma\bar g$. Then, for $L = q^{-1}(\bar L)$ and $g \in G$ any preimage of $\bar g$ under q, μ is the unique L-invariant measure supported on $L\cdot\Gamma g$.*

Proof. By virtue of Lemma A.6, we might assume without loss of generality that $\bar L = \bar G$, so that μ projects to the unique $\bar G$-invariant probability measure $m_{\Gamma\backslash G/R}$ on $\bar\Gamma\backslash\bar G = \Gamma\backslash G/R$. We wish to deduce that $\mu = m_{\Gamma\backslash G}$, the unique G-invariant probability measure on $\Gamma\backslash G$. Let $T\colon \mathrm{C_c}(\Gamma\backslash G) \to \mathrm{C_c}(\Gamma\backslash G/R)$ be as in Lemma A.5. It suffices to prove that

$$\int_{\Gamma\backslash G}\varphi\;\mathrm{d}\mu = \int_{\Gamma\backslash G/R} T\varphi\;\mathrm{d}m_{\Gamma\backslash G/R} \quad \text{for all } \varphi \in \mathrm{C_c}(\Gamma\backslash G); \tag{A.4}$$

if this holds, then by (A.3) μ and $m_{\Gamma\backslash G}$ represent the same positive linear functional on $\mathrm{C_c}(\Gamma\backslash G)$, whence they are equal by uniqueness in the Riesz representation theorem.

Fix thus a function $\varphi \in \mathrm{C_c}(\Gamma\backslash G)$; spelling out the integral on the right-hand side of (A.4), we get

$$\begin{aligned}\int_{\Gamma\backslash G/R} T\varphi\;\mathrm{d}m_{\Gamma\backslash G/R} &= \int_{\Gamma\backslash G/R}\int_{\Gamma\backslash G}\varphi(y)\;\mathrm{d}m_{R\cdot x}(y)\;\mathrm{d}m_{\Gamma\backslash G/R}(\pi(x))\\ &= \int_{\Gamma\backslash G}\int_{\Gamma\backslash G}\varphi(y)\;\mathrm{d}m_{R\cdot x}(y)\;\mathrm{d}\mu(x).\end{aligned} \tag{A.5}$$

where the last equality follows from the assumption $\pi_*\mu = m_{\Gamma\backslash G/R}$. Now, for any $x = \Gamma g \in \Gamma\backslash G$, we use that $m_{R\cdot\Gamma g} = (g^{-1})_* m_{R\cdot\Gamma}$ and write

$$\int_{\Gamma\backslash G} \varphi(y)\, \mathrm{d}m_{R\cdot\Gamma g}(y) = \int_{\Gamma\backslash G} \varphi(g^{-1}\cdot y)\, \mathrm{d}m_{R\cdot\Gamma}(y) = \int_{\Gamma\backslash G} \varphi(g^{-1}\cdot \Gamma r)\, \mathrm{d}m_{R\cdot\Gamma}(\Gamma r).$$

Inserting this into (A.5) and interchanging the integrals via Fubini's theorem, we obtain

$$\begin{aligned}\int_{\Gamma\backslash G/R} T\varphi\, \mathrm{d}m_{\Gamma\backslash G/R} &= \int_{\Gamma\backslash G}\int_{\Gamma\backslash G} \varphi(\Gamma r g)\, \mathrm{d}\mu(\Gamma g)\, \mathrm{d}m_{R\cdot\Gamma}(\Gamma r)\\ &= \int_{\Gamma\backslash G}\int_{\Gamma\backslash G} \varphi(\Gamma g(g^{-1} r g))\, \mathrm{d}\mu(\Gamma g)\, \mathrm{d}m_{R\cdot\Gamma}(\Gamma r)\\ &= \int_{\Gamma\backslash G}\left(\int_{\Gamma\backslash G} \varphi(\Gamma g)\, \mathrm{d}\mu(\Gamma g)\right) \mathrm{d}m_{R\cdot\Gamma}(\Gamma r)\\ &= \int_{\Gamma\backslash G} \varphi\, \mathrm{d}\mu,\end{aligned}$$

where the third equality stems from R-invariance of μ and the fact that R is a normal subgroup of G. This achieves the proof of the proposition. □

Appendix B. Normal subgroups of products

This section is devoted to the proof of Proposition 8.4.

Proof of Proposition 8.4. We wish to show that any Zariski-connected normal subgroup $\mathbf{L} < \mathbf{G}\times\mathbf{B}$, where $\mathbf{G}$ and $\mathbf{B}$ are respectively semisimple and solvable Zariski-connected $\mathbb{Q}$-groups, takes the form $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$, where $\mathbf{L}_1, \mathbf{L}_2$ are Zariski-connected normal $\mathbb{Q}$-subgroups of $\mathbf{G}, \mathbf{B}$ respectively. Let $\mathbf{L} = \mathbf{L}_s \ltimes R_{\mathrm{u}}(\mathbf{L})$ be a Levi decomposition of $\mathbf{L}$, with $\mathbf{L}_s$ a reductive $\mathbb{Q}$-subgroup of $\mathbf{L}$. Since $\mathbf{L}$ is a normal subgroup of $\mathbf{G}\times\mathbf{B}$ and the unipotent radical $R_{\mathrm{u}}(\mathbf{L})$ is a characteristic[21] subgroup of $\mathbf{L}$, we deduce as a result that $R_{\mathrm{u}}(\mathbf{L})$ is a normal subgroup of $\mathbf{G}\times\mathbf{B}$; being also Zariski-connected and unipotent, it is contained in $R_{\mathrm{u}}(\mathbf{G}\times\mathbf{B}) = R_{\mathrm{u}}(\mathbf{B})$, the latter equality holding as $\mathbf{G}$ has trivial unipotent radical. Set thus $\mathbf{L}_2 \coloneqq R_{\mathrm{u}}(\mathbf{L})$.

On the other hand, we claim that the Levi subgroup $\mathbf{L}_s$ coincides with its commutator $[\mathbf{L}_s, \mathbf{L}_s]$ (hence it is semisimple). This is a consequence of the class-$\mathcal{F}$ assumption on $\mathbf{L}$. Indeed, by the structure theorem for reductive subgroups (see, for instance, [53, Thm. 2.4]), $\mathbf{L}_s$ is an almost direct product of a $\mathbb{Q}$-torus $\mathbf{T}$ and of the semisimple group $[\mathbf{L}_s, \mathbf{L}_s]$. If $\mathbf{L}_s \neq [\mathbf{L}_s, \mathbf{L}_s]$, then by the class-$\mathcal{F}$ assumption the group of $\mathbb{Q}_S$-points of the quotient $\mathbf{L}/([\mathbf{L}_s, \mathbf{L}_s]R_u(\mathbf{L})) \simeq \mathbf{L}_s/[\mathbf{L}_s, \mathbf{L}_s] \cong \mathbf{T}$ has non-trivial unipotent elements. This is absurd. The projection of $\mathbf{L}_s$ to the solvable factor $\mathbf{B}$ is thus necessarily trivial, so that $\mathbf{L}_s$ is contained in $\mathbf{G}$ and is a normal subgroup thereof. Setting $\mathbf{L}_1 \coloneqq \mathbf{L}_s$ permits us to conclude. □

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[21] A subgroup R' of an abstract group R is called *characteristic* if it is invariant under any group automorphism of R. In the context of k-algebraic groups, we refer to $\mathbf{R}'$ as a characteristic k-subgroup of a k-group $\mathbf{R}$ if it is invariant under all k-rational group automorphisms of $\mathbf{R}$.

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